UNIVERSIDAD POLITECNICA DE VALENCIA

DEPARTAMENTO DE INGENIERIA MECANICA Y DE MATERIALES

ELEMENTOS FINITOS

(E.T.S.I.I.V)

FORMULACION DE ELEMENTOS FINITOS

LECCION 8.- REQUERIMIENTOS DE CONVERGENCIA

J. L. OLIVER
Dr. Ingeniero Industrial

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001 SIGNIFICADO DEL TERMINO "CONVERGENCIA" CURSO 2004-5

Convergence: discrete (FEM) solution approaches the analytical (math model) solution in some sense

Convergence = Consistency + Stability

(Lax-Wendroff)

001 SIGNIFICADO DE LOS TERMINOS "CONSISTENCIA" Y "ESTABILIDAD" CURSO 2004-5

- **Consistency**

  Completeness *individual elements*

  Compatibility *element patches*

- **Stability**

  Rank Sufficiency *individual elements*

  Positive Jacobian *individual elements*

Completeness. The elements must have enough *approximation power* to capture the analytical solution in the limit of a mesh refinement process.

Compatibility. The shape functions must provide *displacement continuity* between elements.

Completeness and compatibility are two aspects of the so-called **consistency** condition between the discrete and mathematical models.

001 SIGNIFICADO DEL "INDICE VARIACIONAL" CURSO 2004-5

The FEM is based on the direct discretization of an energy functional $\Pi[u]$, where u (displacements for the elements considered in this book) is the primary variable, or (equivalently) the function to be varied. Let m be the highest spatial derivative order of u that appears in Π. This m is called the *variational index*.

Total Potential Energy of Plate in Plane Stress

$$\Pi = U - W$$

$$U = \frac{1}{2}\int_\Omega h\,\boldsymbol{\sigma}^T \mathbf{e}\,d\Omega = \frac{1}{2}\int_\Omega h\,\mathbf{e}^T \mathbf{E}\mathbf{e}\,d\Omega$$

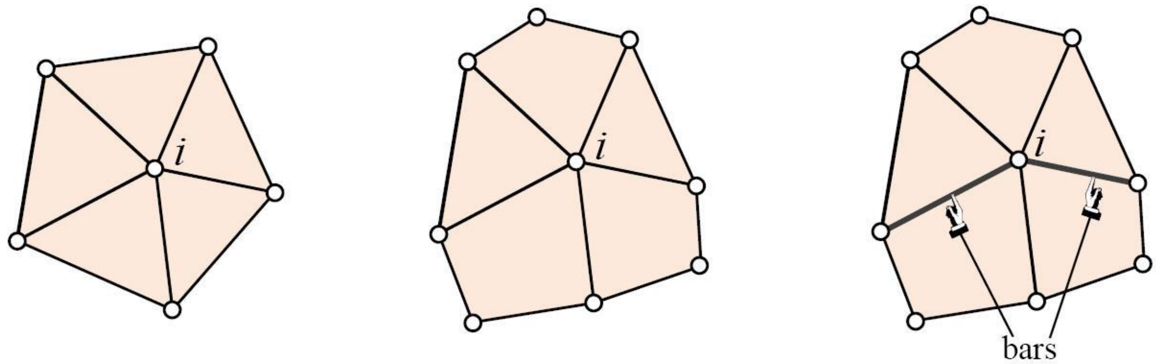
$$W = \int_\Omega h\,\mathbf{u}^T \mathbf{b}\,d\Omega + \int_{\Gamma_t} h\,\mathbf{u}^T \hat{\mathbf{t}}\,d\Gamma$$

$$\mathbf{e}(x,y) = \begin{bmatrix} \frac{\partial N_1^{(e)}}{\partial x} & 0 & \frac{\partial N_2^{(e)}}{\partial x} & 0 & \dots & \frac{\partial N_n^{(e)}}{\partial x} & 0 \\ 0 & \frac{\partial N_1^{(e)}}{\partial y} & 0 & \frac{\partial N_2^{(e)}}{\partial y} & \dots & 0 & \frac{\partial N_n^{(e)}}{\partial y} \\ \frac{\partial N_1^{(e)}}{\partial y} & \frac{\partial N_1^{(e)}}{\partial x} & \frac{\partial N_2^{(e)}}{\partial y} & \frac{\partial N_2^{(e)}}{\partial x} & \dots & \frac{\partial N_n^{(e)}}{\partial y} & \frac{\partial N_n^{(e)}}{\partial x} \end{bmatrix} \mathbf{u}^{(e)} = \mathbf{B}\mathbf{u}^{(e)}$$

Plane Stress: $m = 1$ in **Two Dimensions**

001 DEFINICION DE UN "PATCH" DE ELEMENTOS CURSO 2004-5

A ***patch*** is the set of all elements attached to a given node:



A finite element ***patch trial function*** is the union of shape functions activated by setting a degree of freedom at that node to unity, while all other freedoms are zero. A patch trial function "propagates" only over the patch, and is zero beyond it.

001 1º REQUERIMIENTO FUNDAMENTAL A CUMPLIR POR LAS FUNCIONES DE FORMA, EXPRESADOS EN TERMINOS DEL "INDICE VARIACIONAL" CURSO 2004-5

Completeness

The ***element shape functions*** must represent exactly all polynomial terms of order $\leq m$ in the Cartesian coordinates. A set of shape functions that satisfies this condition is call m-complete

Note that this requirement applies *at the element level* and involves *all* shape functions of the element.

Plane Stress: $m = 1$ in Two Dimensions

Completeness

The ***element shape functions*** must represent exactly all polynomial terms of order ≤ 1 in the Cartesian coordinates. That means any *linear polynomial* in x, y with a *constant* as special case

Suppose a displacement-based element is for a plane stress problem, in which $m = 1$. Then 1-completeness requires that the linear displacement field

$$u_x = \alpha_0 + \alpha_1 x + \alpha_2 y, \qquad u_y = \alpha_0 + \alpha_1 x + \alpha_2 y \tag{19.3}$$

be exactly represented for any value of the α coefficients. This is done by evaluating (19.3) at the nodes to form a displacement vector $\mathbf{u}^{(e)}$ and then checking that $\mathbf{u} = \mathbf{N}^{(e)}\mathbf{u}^{(e)}$ recovers exactly (19.3).

The analysis shows

that completeness is satisfied if the *sum of the shape functions is unity* and *the element is compatible*.

001 2º REQUERIMIENTO FUNDAMENTAL A CUMPLIR POR LAS FUNCIONES DE FORMA, EXPRESADOS EN TERMINOS DEL "INDICE VARIACIONAL" CURSO 2004-5

Compatibility

The ***patch trial functions*** must be $C^{(m-1)}$ continuous between elements, and C^{m} piecewise differentiable inside each element

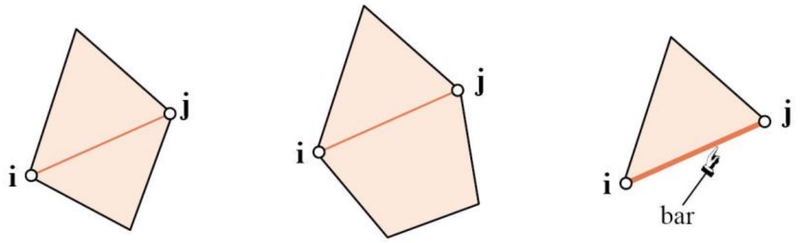
Plane Stress: $m = 1$ in Two Dimensions

Compatibility

The ***patch trial functions*** must be C^{0} continuous between elements, and C^{1} piecewise differentiable inside each element

Interelement Continuity is the Toughest to Meet

Simplification: for *matching meshes* (defined in Notes) it is sufficient to check a *pair of adjacent elements*:



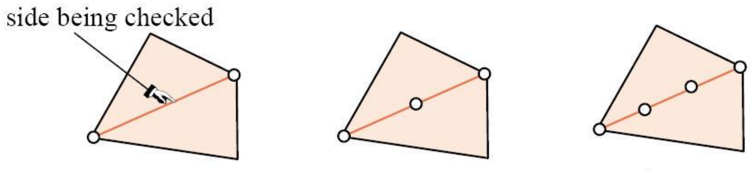
Two 3-node linear triangles

One 3-node linear triangle and one 4-node bilinear quad

One 3-node linear triangle and one 2-node bar

Side Continuity Check for Plane Stress Elements with Polynomial Shape Functions in Natural Coordinates

Let k be the number of nodes on a side:



$k = 2$ $k = 3$ $k = 4$

The variation of each element shape function along the side must be of polynomial order k -1

If ***more, continuity is violated***

If ***less, nodal configuration is wrong*** (too many nodes)

001 1º REQUERIMIENTO A CUMPLIR POR LA MATRIZ DE RIGIDEZ DEL ELEMENTO PARA ASEGURAR LA “ESTABILIDAD” ”SUFICIENCIA DE RANGO” CURSO 2004-5

The element stiffness matrix must not possess any zero-energy kinematic mode other than rigid body modes.

This can be mathematically expressed as follows. Let n_F be the number of element degrees of freedom, and n_R be the number of independent rigid body modes. Let r denote the rank of $\mathbf{K}^{(e)}$. The element is called *rank sufficient* if $r = n_F - n_R$ and *rank deficient* if $r < n_F - n_R$. In the latter case,

$$d = (n_F - n_R) - r \tag{19.5}$$

is called the rank deficiency.

If an isoparametric element is numerically integrated, let n_G be the number of Gauss points, while n_E denotes the order of the stress-strain matrix $\mathbf{E}$. Two additional assumptions are made:

(i) The element shape functions satisfy completeness in the sense that the rigid body modes are exactly captured by them.

(ii) Matrix $\mathbf{E}$ is of full rank.

Then each Gauss point adds n_E to the rank of $\mathbf{K}^{(e)}$, up to a maximum of $n_F - n_R$. Hence the rank of $\mathbf{K}^{(e)}$ will be

$$r = \min(n_F - n_R, n_E\, n_G) \tag{19.6}$$

To attain rank sufficiency, $n_E n_G$ must equal or exceed $n_F - n_R$:

$$\boxed{n_E\, n_G \geq n_F - n_R} \tag{19.7}$$

from which the appropriate Gauss integration rule can be selected.

In the plane stress problem, $n_E = 3$ because $\mathbf{E}$ is a 3×3 matrix of elastic moduli; see Chapter 14. Also $n_R = 3$. Consequently $r = \min(n_F - 3, 3n_G)$ and $3n_G \geq n_F - 3$.

EXAMPLE 19.5

Consider a plane stress 6-node quadratic triangle. Then $n_F = 2 \times 6 = 12$. To attain the proper rank of $12 - n_R = 12 - 3 = 9$, $n_G \geq 3$. A 3-point Gauss rule, such as the midpoint rule defined in §24.2, makes the element rank sufficient.

EXAMPLE 19.6

Consider a plane stress 9-node biquadratic quadrilateral. Then $n_F = 2 \times 9 = 18$. To attain the proper rank of $18 - n_R = 18 - 3 = 15$, $n_G \geq 5$. The 2×2 product Gauss rule is insufficient because $n_G = 4$. Hence a 3×3 rule, which yields $n_G = 9$, is required to attain rank sufficiency.

Table 19.1 collects rank-sufficient Gauss integration rules for some widely used plane stress elements with n nodes and $n_F = 2n$ freedoms.

Element	n	n_F	$n_F - 3$	Min n_G	Recommended rule
3-node triangle	3	6	3	1	centroid*
6-node triangle	6	12	9	3	3-midpoint rule*
10-node triangle	10	20	17	6	7-point rule*
4-node quadrilateral	4	8	5	2	2 x 2
8-node quadrilateral	8	16	13	5	3 x 3
9-node quadrilateral	9	18	15	5	3 x 3
16-node quadrilateral	16	32	29	10	4 x 4

001 2º REQUERIMIENTO A CUMPLIR POR LA GEOMETRIA DEL ELEMENTO PARA ASEGURAR LA "ESTABILIDAD" "JACOBIANO POSITIVO" **CURSO 2004-5**

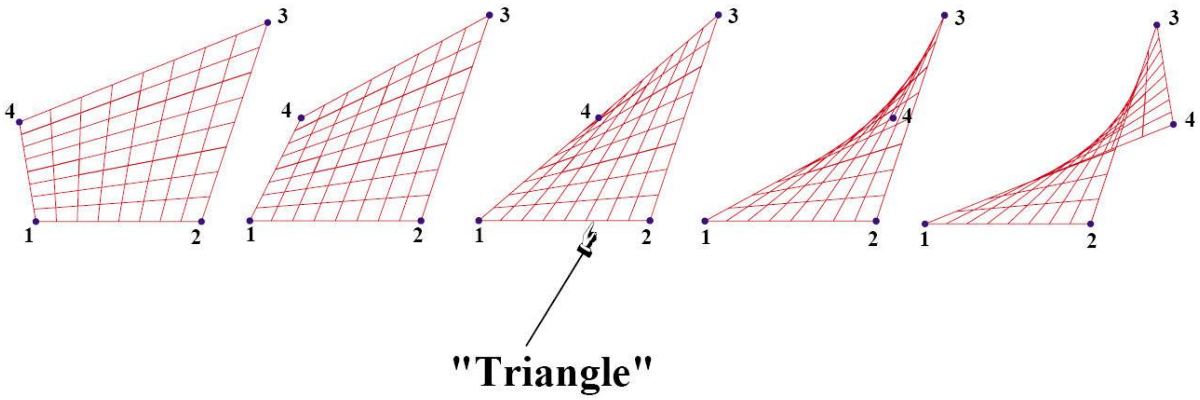
The geometry of the element must be such that the determinant $J = \det \mathbf{J}$ of the Jacobian matrix defined[4] in §17.2, is positive everywhere. As illustrated in Equation (17.20), J characterizes the local metric of the element natural coordinates.

001 TRIANGULO DE 3 NODOS **CURSO 2004-5**

For a three-node triangle J is constant and in fact equal to $2A$. The requirement $J > 0$ is equivalent to saying that corner nodes must be positioned and numbered so that a positive area $A > 0$ results. This is called a *convexity condition*. It is easily checked by a finite element program.

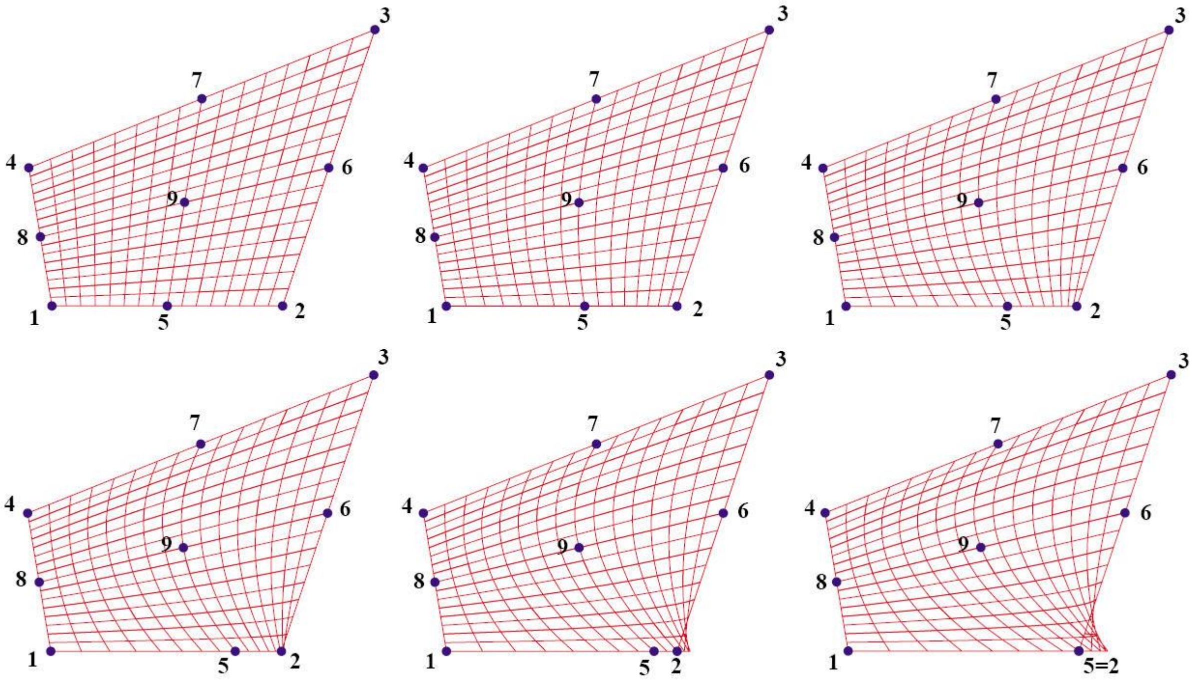
001 CUADRILATERO DE 4 NODOS **CURSO 2004-5**

But for 2D elements with more than 3 nodes distortions may render *portions* of the element metric negative. This is illustrated in Figure 19.2 for a 4-node quadrilateral in which node 4 is gradually moved to the right. The quadrilateral morphs from a convex figure into a nonconvex one. The center figure is a triangle; note that the metric near node 4 is badly distorted (in fact $J = 0$ there) rendering the element unacceptable. This contradicts the (erroneous) advise of some FE books, which state that quadrilaterals can be reduced to triangles as special cases, thereby rendering triangular elements unnecessary.



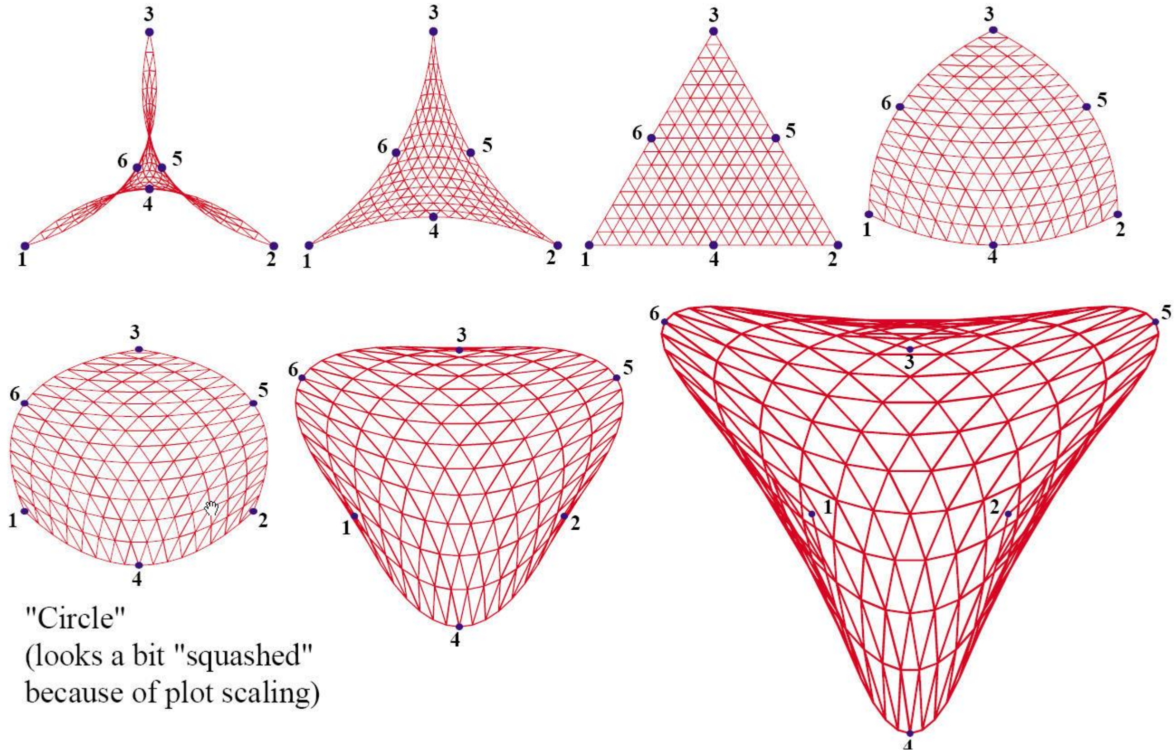
For higher order elements proper location of corner nodes is not enough. The non-corner nodes (midside, interior, etc.) must be placed sufficiently close to their natural locations (midpoints, centroids, etc.) to avoid violent local distortions. The effect of midpoint motions in quadratic elements is illustrated in Figures 19.3 and 19.4.

Figure 19.3 depicts the effect of moving midside node 5 tangentially in a 9-node quadrilateral element while keeping all other 8 nodes fixed. When the location of 5 reaches the quarter-point of side 1-2, the metric at corner 2 becomes singular in the sense that $J = 0$ there. Although this is disastrous in ordinary FE work, it has applications in the construction of special "crack" elements for linear fracture mechanics.



001	TRIANGULO DE SEIS NODOS	CURSO 2004-5

Displacing midside nodes normally to the sides is comparatively more forgiving, as illustrated in Figure 19.4. This depicts a 6-node equilateral triangle in which midside nodes 4, 5 and 6 are moved inwards and outwards along the normals to the midpoint location. As shown in the lower left picture, the element may be even morphed into a "parabolic circle" without the metric breaking down.



001	EJERCICIO 1	CURSO 2004-5

EXERCISE 19.1

[D:15] Draw a picture of a 2D non-matching mesh in which element nodes on two sides of a boundary do not share the same locations. Discuss why enforcing compatibility becomes difficult.

001	EJERCICIO 2	CURSO 2004-5

EXERCISE 19.4

[A:20] Consider three dimensional solid "brick" elements with n nodes and 3 degrees of freedom per node so $n_F = 3n$. The correct number of rigid body modes is 6. Each Gauss integration point adds 6 to the rank; that is, $N_E = 6$. By applying (19.7), find the minimal rank-preserving Gauss integration rules with p points in each direction (that is, $1\times1\times1$, $2\times2\times2$, etc) if the number of node points is n = 8, 20, 27, or 64.