UNIVERSIDAD POLITECNICA DE VALENCIA

DEPARTAMENTO DE INGENIERIA MECANICA Y DE MATERIALES

ELEMENTOS FINITOS

(E.T.S.I.I.V)

FORMULACION DE ELEMENTOS FINITOS

LECCION 7.- FUNCIONES DE FORMA. SU “MAGIA”

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Dr. Ingeniero Industrial

Valencia, 2005

'Magic' Means *Direct*

Do in 15 minutes what took smart people several months (and less gifted, several years)

But ... it looks like magic to the uninitiated

Shape Function Requirements

(A) Interpolation

(B) Local Support

(C) Continuity (Intra- & Inter-Element)

(D) Completeness

(A) *Interpolation condition.* Takes a unit value at node i, and is zero at all other nodes.

(B) *Local support condition.* Vanishes along any element boundary (a side in 2D, a face in 3D) that does not include node i.

(C) *Interelement compatibility condition.* Satisfies C^0 continuity between adjacent elements on any element boundary that includes node i.

(D) *Completeness condition.* The interpolation is able to represent exactly any displacement field which is a linear polynomial in x and y; in particular, a constant value.

A statement equivalent to (C) is that the value of the shape function along a side common to two elements must uniquely depend only on its nodal values on that side.

Completeness is a property of *all* element isoparametric shape functions taken together, rather than of an individual one. If the element satisfies (B) and (C), in view of the discussion in §16.6 it is sufficient to check that the *sum of shape functions is identically one*.

001 CONSTRUCCION DIRECTA DE FUNCIONES DE FORMA CURSO 2004-5

Direct Construction of Shape Functions as "Line Products"

$$N_i^{(e)} \stackrel{\text{guess}}{=} c_i \, L_1 \, L_2 \, \ldots \, L_m$$

where $L_k = 0$ are equations of "lines" expressed in natural coordinates, that cross *all nodes* except i

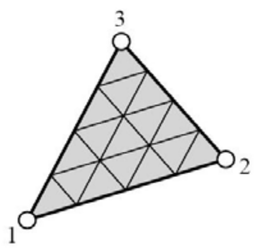
For two-dimensional isoparametric elements, the ingredients in (18.1) are chosen according to the following rules.

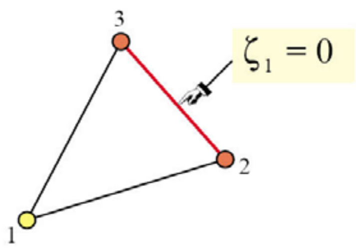
1. Select the L_j as the minimal number of lines or curves linear in the natural coordinates that cross all nodes except the i^{th} node. Primary choices are the element sides and medians. The examples below illustrate how this is done.
2. Set coefficient c_i so that $N_i^{(e)}$ has the value 1 at the i^{th} node.
3. Check the polynomial order variation over each interelement boundary that contains node i. If this order is n, there must be exactly $n + 1$ nodes on the boundary for the compatibility condition to hold.
4. If compatibility is satisfied, check that the sum of shape functions is identically one.

Specific two-dimensional examples in the following subsections show these rules in action. Essentially the same technique is applicable to one- and three-dimensional elements.

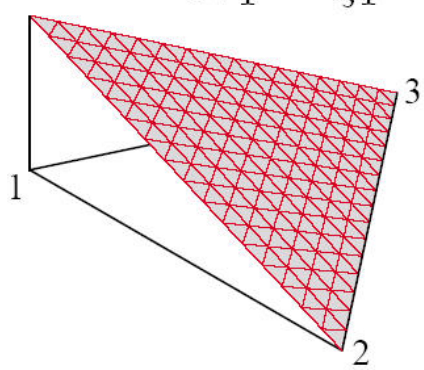
001 APLICACIÓN AL TRIANGULO DE TRES NODOS LINEAL CURSO 2004-5

The Three Node Linear Triangle





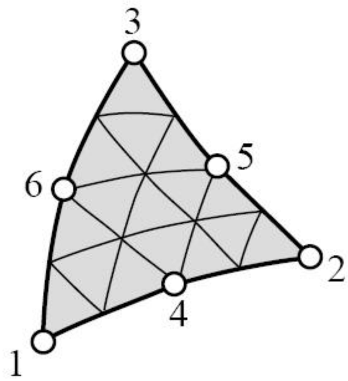
$$N_1 = \zeta_1$$

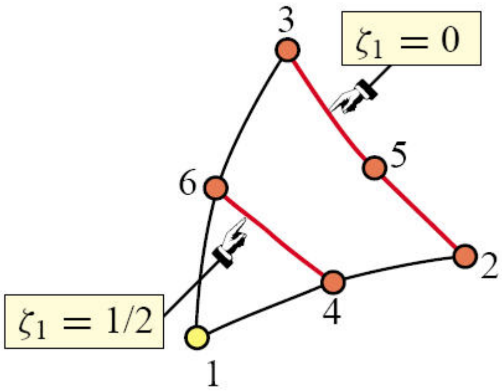


$$N_1 \stackrel{\text{guess}}{=} c_1 L_1 = c_1 L_{2\text{-}3}$$

At node 1, $N_1 = 1$ whence $c_1 = 1$
and $N_1 = \zeta_1$ Likewise for N_2 and N_3

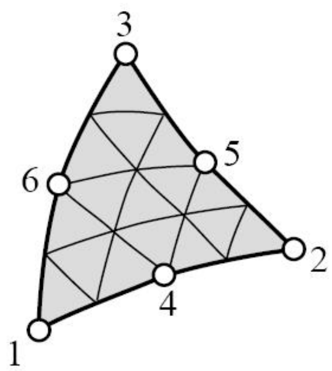
001 **TRIANGULO SEIS NODOS - NODO ESQUINA** CURSO 2004-5

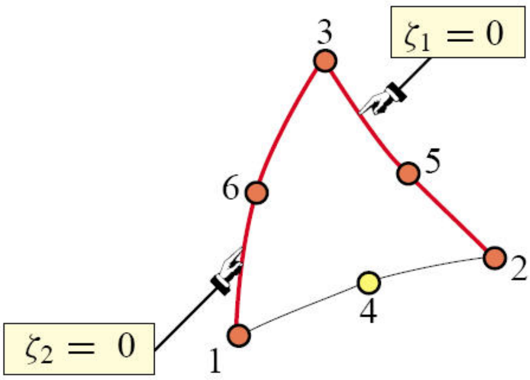




$$N_1^{(e)} \overset{\text{guess}}{=} c_1\, L_{2\text{-}3}\, L_{4\text{-}6}$$

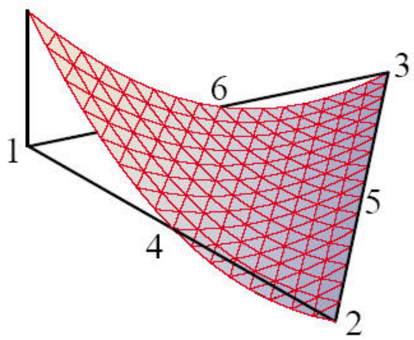
001 **TRIANGULO SEIS NODOS - NODO MEDIO** CURSO 2004-5

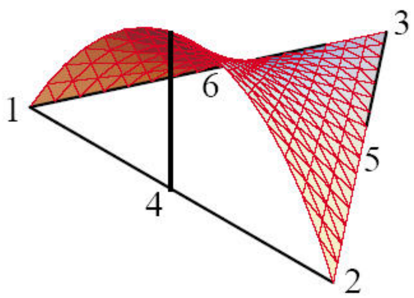




$$N_1^{(e)} \overset{\text{guess}}{=} c_1\, L_{2\text{-}3}\, L_{4\text{-}6}$$

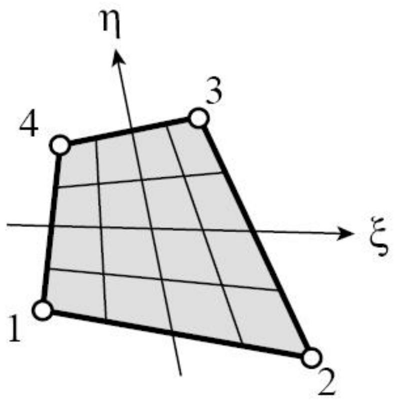
001 **GRAFICO FUNCIONES FORMA** CURSO 2004-5

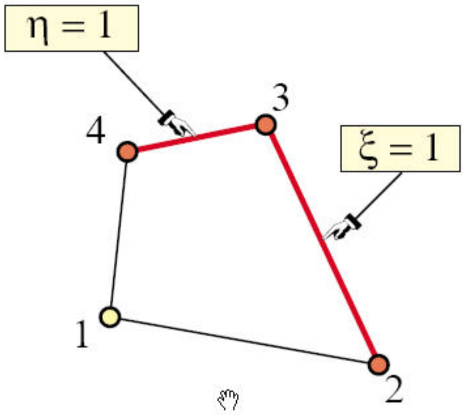




$$N_1^{(e)} = \zeta_1(2\zeta_1 - 1) \qquad N_4^{(e)} = 4\zeta_1\zeta_2$$

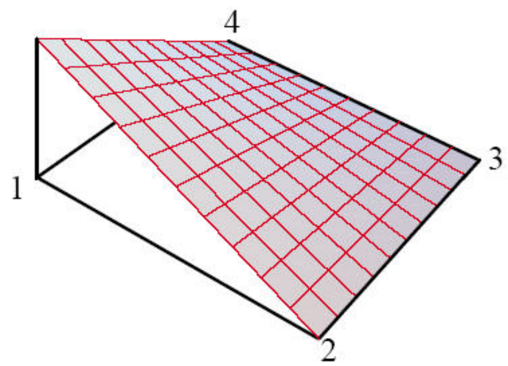
001 **CUADRILATERO BILINEAL DE CUATRO NODOS** CURSO 2004-5





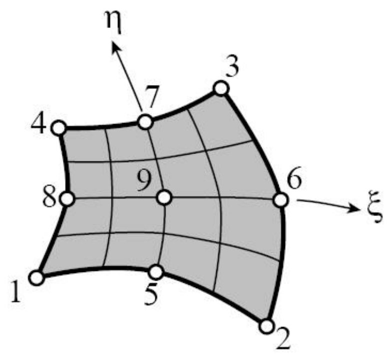
$$N_1^{(e)} \overset{\text{guess}}{=} c_1\, L_{2\text{-}3}\, L_{3\text{-}4}$$

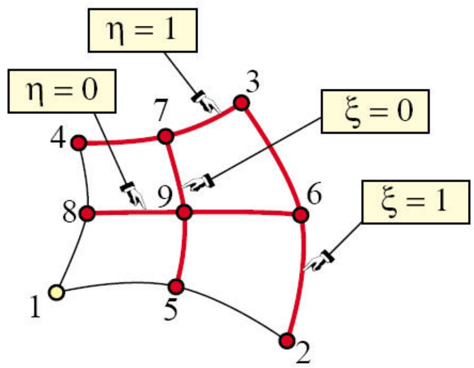
001 **GRAFICO FUNCIONES FORMA** CURSO 2004-5



$$N_1^{(e)} = \frac{1}{4}(1-\xi)(1-\eta)$$

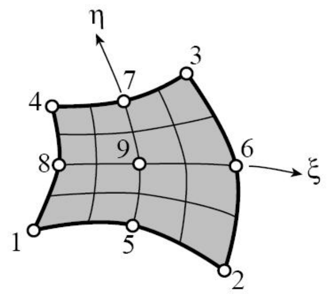
001 **CUADRILATERO BICUADRATICO DE NUEVE NODOS - NODO ESQUINA** CURSO 2004-5

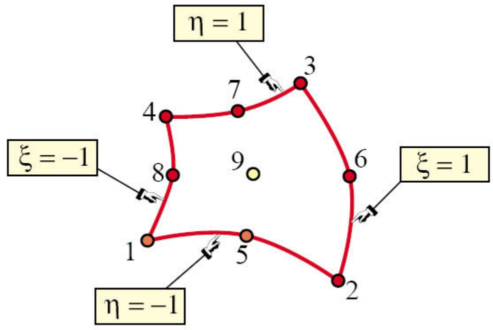




$$N_1^{(e)} \overset{\text{guess}}{=} c_1 L_{2\text{-}3} L_{3\text{-}4} L_{5\text{-}7} L_{6\text{-}8} = c_1(\xi-1)(\eta-1)\xi\eta$$

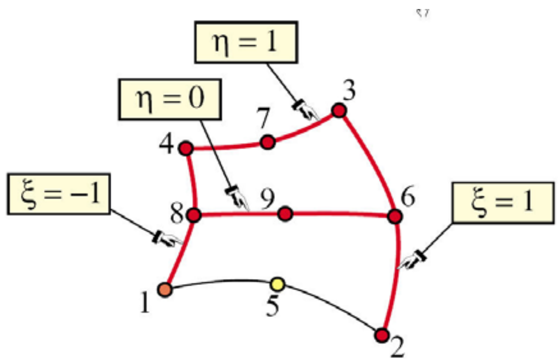
001 **CUADRILATERO BICUADRATICO DE NUEVE NODOS – NODO INTERNO** CURSO 2004-5





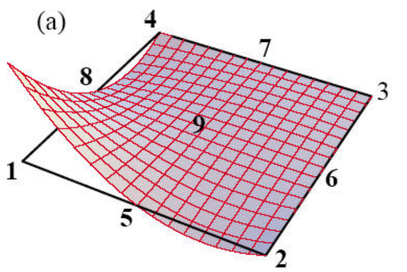
$$N_9^{(e)} = c_9\, L_{1\text{-}2} L_{2\text{-}3} L_{3\text{-}4} L_{4\text{-}1} = c_9\,(\xi-1)(\eta-1)(\xi+1)(\eta+1)$$

001 **CUADRILATERO BICUADRATICO DE NUEVE NODOS – NODO MEDIO** CURSO 2004-5

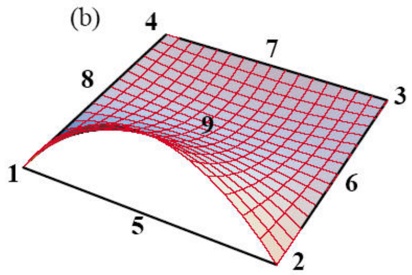


$$N_5^{(e)} = c_5 L_{2\text{-}3} L_{1\text{-}4} L_{6\text{-}8} L_{3\text{-}4} = c_5\,(\xi-1)(\xi+1)\eta(\eta-1) = c_5\,(1-\xi^2)\eta(1-\eta)$$

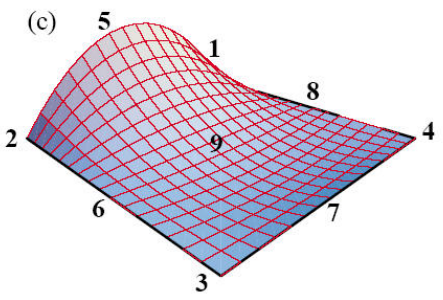
001 **GRAFICO FUNCIONES FORMA** CURSO 2004-5



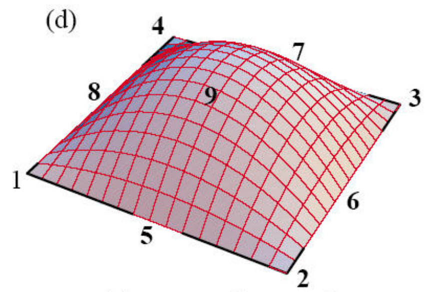
$N_1^{(e)} = \frac{1}{4}(\xi-1)(\eta-1)\xi\eta$



$N_5^{(e)} = \frac{1}{2}(1-\xi^2)\eta(\eta-1)$

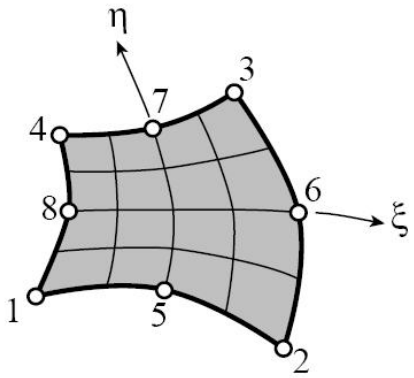


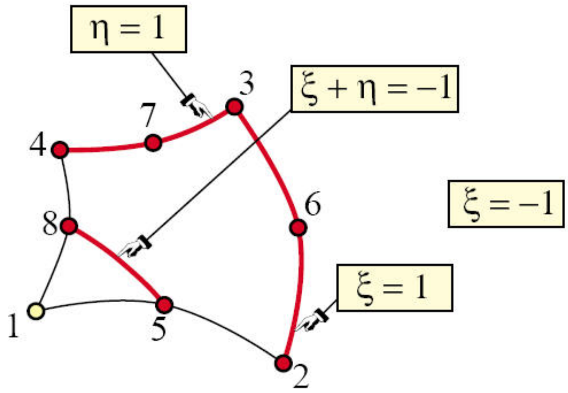
$N_5^{(e)} = \frac{1}{2}(1-\xi^2)\eta(\eta-1)$ (back view)



$N_9^{(e)} = (1-\xi^2)(1-\eta^2)$

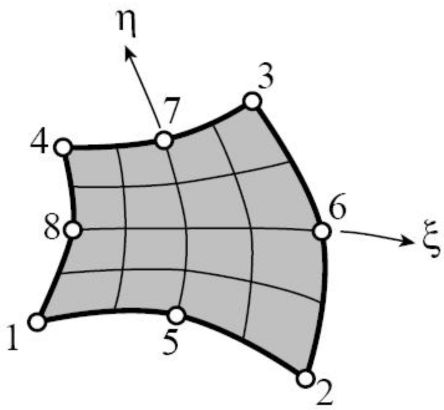
001 CUADRILATERO “SERENDIPITO” DE OCHO NODOS – NODO INTERNO CURSO 2004-5





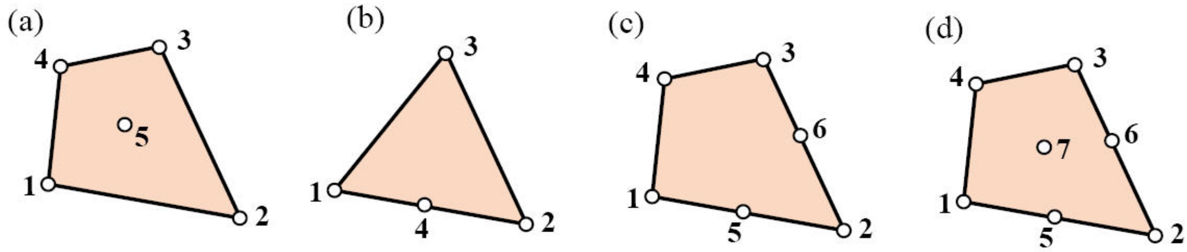
$$N_1^{(e)} = c_1 L_{2\text{-}3} L_{3\text{-}4} L_{5\text{-}8} = c_1(\xi - 1)(\eta - 1)(1 + \xi + \eta)$$

001 CUADRILATERO “SERENDIPITO” DE OCHO NODOS – NODO MEDIO CURSO 2004-5

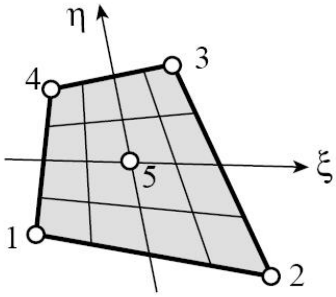


$$N_5^{(e)} = c_5 L_{2\text{-}3} L_{3\text{-}4} L_{4\text{-}1} = c_5\,(\xi - 1)(\xi + 1)(\eta - 1) = c_5\,(1 - \xi^2)(1 - \eta).$$

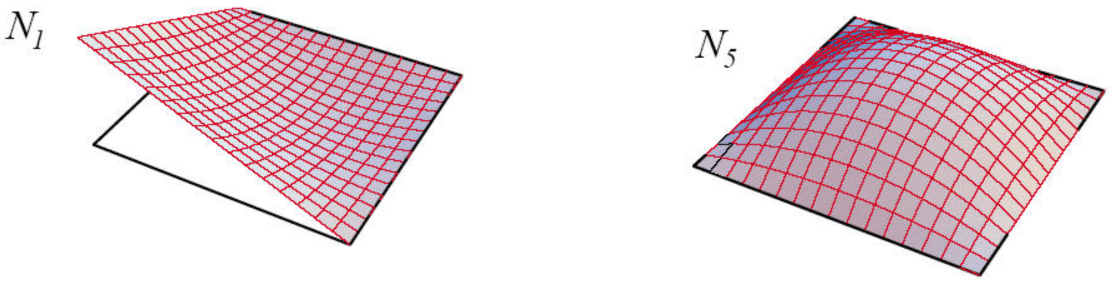
001 A VECES LA TECNICA FALLA, PERO SIGUE SIENDO UN BUEN PUNTO DE PARTIDA CURSO 2004-5

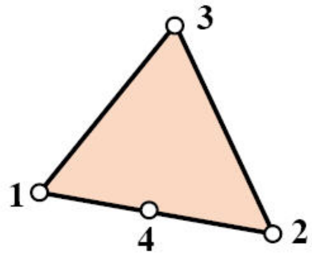


$$N_i^{(e)} = c_i\, L_1^c\, L_2^c\, \ldots\, L_m^c + d_i\, L_1^d\, L_2^d\, \ldots\, L_n^d$$



$$N_1 = -\tfrac{1}{8}(1-\xi)(1-\eta)(\xi+\eta)$$





For N_1 try the magic wand: product of side 2-3 ($\zeta_1 = 0$) and median 3-4 ($\zeta_1 = \zeta_2$):

$$N_1^{(e)} \overset{\text{guess}}{=} c_1\zeta_1(\zeta_1 - \zeta_2), \quad N_1(1,0,0) = 1 = c_1 \qquad \textit{fails (C)}$$

Next, try the shape function of the linear 3-node triangle plus a correction: $N_1^{(e)} \overset{\text{guess}}{=} \zeta_1 + c_1\zeta_1\zeta_2$

Coefficient c_1 is determined by requiring this shape function vanish at midside node 4: $N_1^{(e)}(\tfrac{1}{2}, \tfrac{1}{2}, 0) = \tfrac{1}{2} + c_1\tfrac{1}{4} = 0$, whence $c_1 = -2$ and

$$\boxed{N_1^{(e)} = \zeta_1 - 2\zeta_1\zeta_2} \qquad \textit{works}$$

001 **EJERCICIO 1** **CURSO 2004-5**

EXERCISE 18.1

[A/C:15+10] The complete cubic triangle for plane stress has 10 nodes located as shown in Figure E18.1, with their triangular coordinates listed in parentheses.

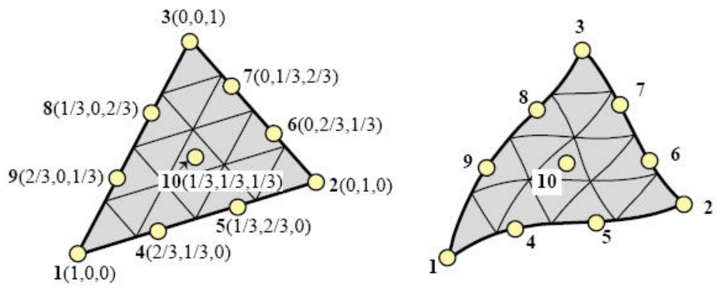


Figure E18.1. Ten-node cubic triangle for Exercise 18.1. The left picture displays the superparametric element whereas the right picture shows the general isoparametric version with curved sides.

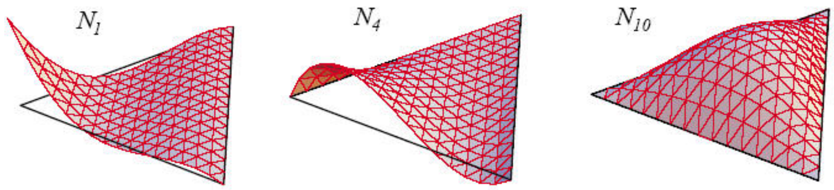


Figure E18.2. Perspective plots of the shape functions N_1, N_4 and N_{10} for the 10-node cubic triangle.

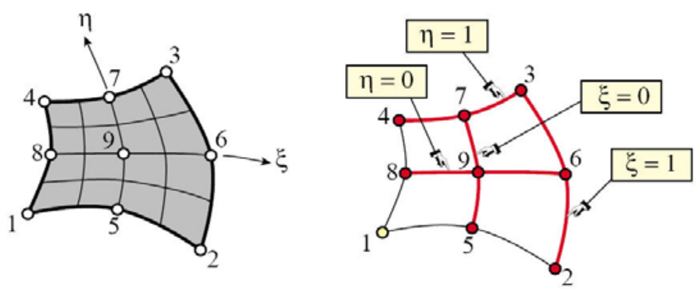
(a) Construct the cubic shape functions $N_1^{(e)}$, $N_4^{(e)}$ and $N_{10}^{(e)}$ for nodes 1, 4, and 10 using the line-product technique. [Hint: each shape function is the product of 3 and only 3 lines.] Perspective plots of those 3 functions are shown in Figure E18.2.

(b) Construct the missing 7 shape functions by appropriate node number permutations, and verify that the sum of the 10 functions is identically one. For the unit sum check use the fact that $\zeta_1 + \zeta_2 + \zeta - 3 = 1$.

001 **EJERCICIO 2** **CURSO 2004-5**

EXERCISE 18.2

[A:15] Find an alternative shape function $N_1^{(e)}$ for corner node 1 of the 9-node quadrilateral of Figure 18.5(a) by using the diagonal lines 5–8 and 2–9–4 in addition to the sides 2–3 and 3–4. Show that the resulting shape function violates the compatibility condition (C) stated in §18.1.



$$N_1^{(e)} \overset{\text{guess}}{=} c_1 L_{2\text{-}3} L_{3\text{-}4} L_{5\text{-}7} L_{6\text{-}8} = c_1(\xi - 1)(\eta - 1)\xi\eta$$

001 **EJERCICIO 3** **CURSO 2004-5**

EXERCISE 18.3

[A/C:15] Complete the above exercise for all nine nodes. Add the shape functions (use a CAS and simplify) and verify whether their sum is unity.

001 **EJERCICIO 4** CURSO 2004-5

EXERCISE 18.4

[A/C:20] Verify that the shape functions $N_1^{(e)}$ and $N_5^{(e)}$ of the eight-node serendipity quadrilateral discussed in §18.4.3 satisfy the interelement compatibility condition (C) stated in §18.1. Obtain all 8 shape functions and verify that their sum is unity.

001 **EJERCICIO 5** CURSO 2004-5

EXERCISE 18.5

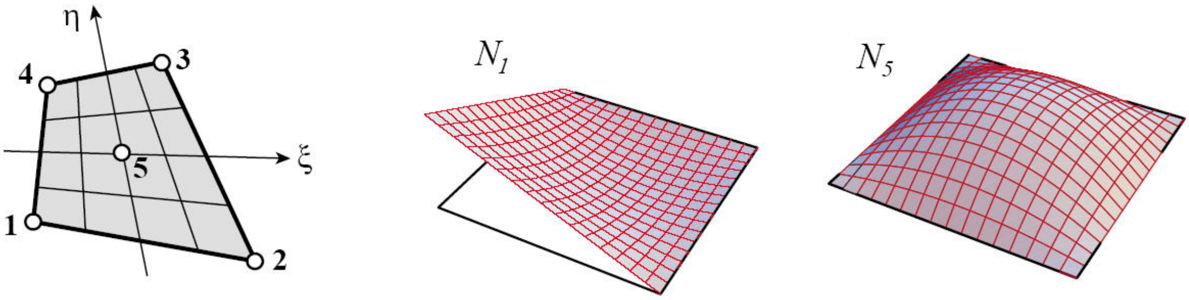
[C:15] Plot the shape functions $N_1^{(e)}$ and $N_5^{(e)}$ of the eight-node serendipity quadrilateral studied in §18.4.3 using the module `PlotQuadrilateralShapeFunction` listed in Cell 18.2.

001 **EJERCICIO 6** CURSO 2004-5

EXERCISE 18.6

[A:15] A five node quadrilateral element has the nodal configuration shown in Figure E18.3. Perspective views of $N_1^{(e)}$ and $N_5^{(e)}$ are shown in that Figure.[2] Find five shape functions $N_i^{(e)}$, $i = 1, 2, 3, 4, 5$ that satisfy compatibility, and also verify that their sum is unity.

Hint: develop $N_5(\xi, \eta)$ first for the 5-node quad using the line-product method; then the corner shape functions $\bar{N}_i(\xi, \eta)$ $(i = 1, 2, 3, 4)$ for the 4-node quad (already given in the Notes); finally combine $N_i = \bar{N}_i + \alpha N_5$, determining α so that all N_i vanish at node 5. Check that $N_1 + N_2 + N_3 + N_4 + N_5 = 1$ identically.



[2] Although this $N_1^{(e)}$ resembles the $N_1^{(e)}$ of the 4-node quadrilateral depicted in Figure 18.4, they are not the same. That shown in Figure E18.3 must vanish at node 5, that is, at $\xi = \eta = 0$. On the other hand, the $N_1^{(e)}$ of Figure 18.4 takes the value $\frac{1}{4}$ there.

001 **EJERCICIO 7** CURSO 2004-5

EXERCISE 18.7

[A:15] An eight-node "brick" finite element for three dimensional analysis has three isoparametric natural coordinates called ξ, η and μ. These coordinates vary from -1 at one face to $+1$ at the opposite face, as sketched in Figure E18.4.

Construct the (trilinear) shape function for node 1 (follow the node numbering of the figure). The equations of the brick faces are:

$1485: \ \xi = -1$	$2376: \ \xi = +1$
$1265: \ \eta = -1$	$4378: \ \eta = +1$
$1234: \ \mu = -1$	$5678: \ \mu = +1$

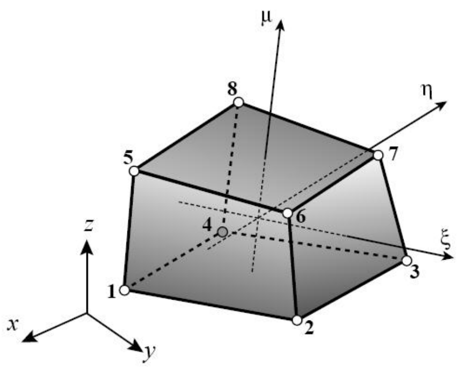
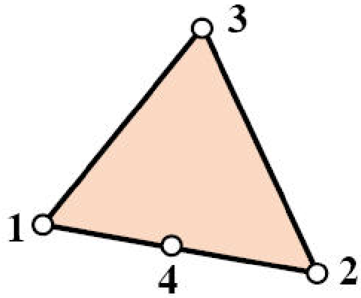


Figure E18.4. Eight-node isoparametric "brick" element for Exercise 18.7.

001 **EJERCICIO 8** CURSO 2004-5

EXERCISE 18.8

[A:15] Consider the 4-node transition triangular element of Figure 18.8(b). The shape function for node 1, $N_1 = \zeta_1 - 2\zeta_1\zeta_2$ was derived in §18.6.2. Show that the others are $N_2 = \zeta_2 - 2\zeta_1\zeta_2$, $N_3 = \zeta_3$ and $N_4 = 4\zeta_1\zeta_2$. Check that compatibility and completeness are verified.



001 **EJERCICIO 9** CURSO 2004-5

EXERCISE 18.9

[A:20] Construct the six shape functions for the 6-node transition quadrilateral element of Figure 18.8(c). Hint: for the corner nodes, use two corrections to the shape functions of the 4-node bilinear quadrilateral. Check compatibility and completeness. Partial result: $N_1 = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta)$.