7.2. Cuadriláteros de Cuatro Nodos para Tensión Plana – Mathematica.

Partiendo de la lectura del capítulo 17 del Prof. Carlos A. Felippa, en esta sección elaboramos un documento de “Mathematica” definiendo el elemento que se presenta en este capítulo, utilizándolo para obtener la Matriz de Rigidez utilizando integración numérica.

CHAPTER 17. Cuadriláteros Isoparamétricos.
Carlos A. Felippa.

ELEMENTO CUADRILATERO LINEAR - 4 NODOS
Elemento Cuadrilatero Bilineal
v.2011

INICIO.

- Directorio de Trabajo e Inicio.

```
SetDirectory["G:\\CUADRILATERO_04"]
```

```
G:\CUADRILATERO_04
```

```
Off[General::spell1]
Off[General::spell]
```

- Referencia.

```
Lección = Import["001.jpg"];
Show[Lección, ImageSize → 300]
```

17

Isoparametric Quadrilaterals

- Referencia.

```
Show[Import["0003.jpg"], ImageSize → 1000]
```

§17.1. Introduction

In this Chapter the isoparametric representation of element geometry and shape functions discussed in the previous Chapter is used to construct *quadrilateral* elements for the plane stress problem. Formulas given in Chapter 14 for the stiffness matrix and consistent load vector of general plane stress elements are of course applicable to these elements. For a practical implementation, however, we must go through more specific steps:

1. Construction of shape functions.
2. Computations of shape function derivatives to form the strain-displacement matrix.
3. Numerical integration over the element by Gauss quadrature rules.

The first topic was dealt in the previous Chapter in recipe form, and is systematically covered in the next one. Assuming the shape functions have been constructed (or readily found in the FEM literature) the second and third items are combined in an algorithm suitable for programming any isoparametric quadrilateral. The implementation of the algorithm in the form of element modules is partly explained in the Exercises of this Chapter, and covered more systematically in Chapter 23.

We shall not deal with isoparametric triangles here to keep the exposition focused. Triangular coordinates, being linked by a constraint, require "special handling" techniques that would complicate and confuse the exposition. Chapter 24 discusses isoparametric triangular elements in detail.

GEOMETRIA DEL CUADRILATERO Y SISTEMA DE COORDENADAS

- **Coordenadas Naturales.**

In[25]:=
```
Coordenadas = Import["002.jpg"];
Show[Coordenadas, ImageSize → 350]
```

Out[26]=

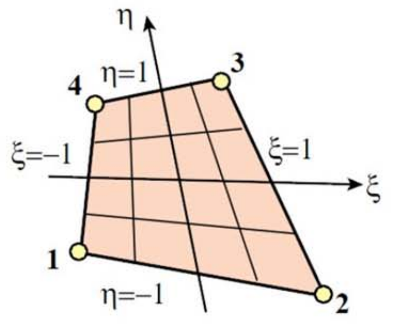
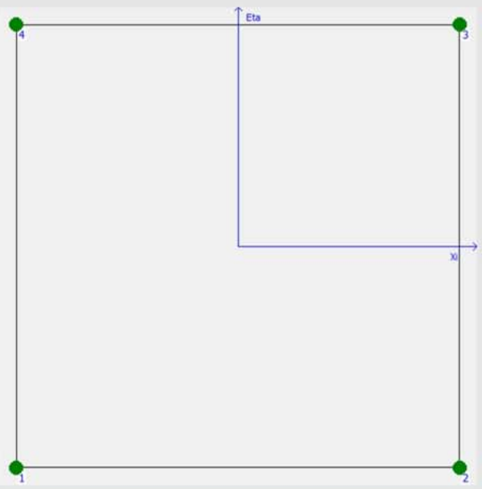


Figure 16.7. The 4-node bilinear quadrilateral.

```
Show[Import["CR4N.bmp"], ImageSize → 300]
```



- Referencia.

In[16]:=
```
Show[Import["0001.jpg"], ImageSize → 1000]
```

Out[16]=

§16.5.1. Quadrilateral Coordinates and Iso-P Mappings

Before presenting examples of quadrilateral elements, we must introduce the appropriate *natural coordinate system* for that geometry. The natural coordinates for a triangular element are the triangular coordinates ζ_1, ζ_2 and ζ_3. The natural coordinates for a quadrilateral element are ξ and η, which are illustrated in Figure 16.6 for both straight sided and curved side quadrilaterals. These are called *quadrilateral coordinates*.

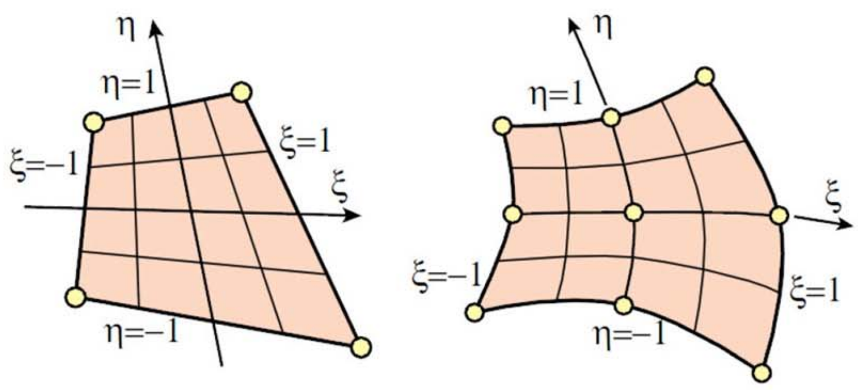


FIGURE 16.6. Quadrilateral coordinates.

These coordinates vary from -1 on one side to $+1$ at the other, taking the value zero over the quadrilateral medians. This particular variation range (instead of taking, say, 0 to 1) was chosen by Irons and coworkers to facilitate use of the standard Gauss integration formulas. Those formulas are discussed in the next Chapter.

Remark 16.2. In some FEM derivations it is convenient to visualize the quadrilateral coordinates plotted as Cartesian coordinates in the $\{\xi, \eta\}$ plane. This is called the *reference plane*. All quadrilateral elements in the reference plane become a square of side 2, called the *reference element*, which extends over $\xi \in [-1, 1]$, $\eta \in [-1, 1]$. The transformation between $\{\xi, \eta\}$ and $\{x, y\}$ dictated by the second and third equations of (16.4), is called the *isoparametric mapping*. A similar version exists for triangles. An important application of this mapping is discussed in §16.6; see Figure 16.9 there.

FORMULACION DE ELEMENTO

▪ Funciones de Forma - Cuadrilatero Bilineal de 4 Nodos - Funciones de Interpolación

In[38]:=
```
FuncionesForma = Import["003.jpg"];
Show[FuncionesForma, ImageSize → 750]
```

Out[39]=

$$N_1^{(e)} = \tfrac{1}{4}(1-\xi)(1-\eta), \qquad N_2^{(e)} = \tfrac{1}{4}(1+\xi)(1-\eta), \qquad (16.13)$$
$$N_3^{(e)} = \tfrac{1}{4}(1+\xi)(1+\eta), \qquad N_4^{(e)} = \tfrac{1}{4}(1-\xi)(1+\eta).$$

$$N1 = \frac{1}{4}(1-\xi) * (1-\eta);$$
$$N2 = \frac{1}{4}(1+\xi) * (1-\eta);$$
$$N3 = \frac{1}{4}(1+\xi) * (1+\eta);$$
$$N4 = \frac{1}{4}(1-\xi) * (1+\eta);$$

Definición del Vector de las Funciones de Forma

$$Nf = \begin{pmatrix} N1 \\ N2 \\ N3 \\ N4 \end{pmatrix};$$

---> FUNCIONES DE FORMA - Nf

```
Nfm = {N1, N2, N3, N4}
```

$$\left\{\frac{1}{4}(1-\eta)(1-\xi), \frac{1}{4}(1-\eta)(1+\xi), \frac{1}{4}(1+\eta)(1+\xi), \frac{1}{4}(1+\eta)(1-\xi)\right\}$$

```
MatrixForm[Nfm]
```

$$\begin{pmatrix} \frac{1}{4}(1-\eta)(1-\xi) \\ \frac{1}{4}(1-\eta)(1+\xi) \\ \frac{1}{4}(1+\eta)(1+\xi) \\ \frac{1}{4}(1+\eta)(1-\xi) \end{pmatrix}$$

```
Nfm
```

$$\left\{\frac{1}{4}(1-\eta)(1-\xi), \frac{1}{4}(1-\eta)(1+\xi), \frac{1}{4}(1+\eta)(1+\xi), \frac{1}{4}(1+\eta)(1-\xi)\right\}$$

```
NNodos = 4;
```

$$Suma = \sum_{i=1}^{NNodos} Nfm[[i]]$$

$$\frac{1}{4}(1-\eta)(1-\xi) + \frac{1}{4}(1+\eta)(1-\xi) + \frac{1}{4}(1-\eta)(1+\xi) + \frac{1}{4}(1+\eta)(1+\xi)$$

```
Simplify[%]
```

1

▫ **Referencia.**

In[17]:=
```
Show[Import["0002.jpg"], ImageSize → 1000]
```

Out[17]=

§16.5.2. The Bilinear Quadrilateral

The four-node quadrilateral shown in Figure 16.7 is the simplest member of the quadrilateral family. It is defined by

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \end{bmatrix}. \qquad (16.12)$$

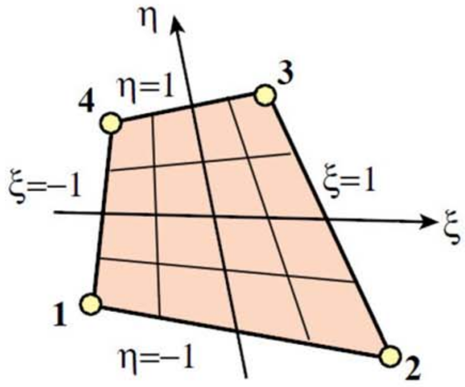


FIGURE 16.7. The 4-node bilinear quadrilateral.

The shape functions are

$$N_1^e = \tfrac{1}{4}(1-\xi)(1-\eta), \qquad N_2^e = \tfrac{1}{4}(1+\xi)(1-\eta),$$
$$N_3^e = \tfrac{1}{4}(1+\xi)(1+\eta), \qquad N_4^e = \tfrac{1}{4}(1-\xi)(1+\eta). \qquad (16.13)$$

These functions vary *linearly* on quadrilateral coordinate lines $\xi = const$ and $\eta = const$, but are not linear polynomials as in the case of the three-node triangle.

▪ **Definicion Isoparamétrica.**

▫ **Definicion Matricial.**

In[42]:=
```
Isoparametrica = Import["004.jpg"];
Show[Isoparametrica, ImageSize → 550]
```

Out[43]=

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \\ N_4^{(e)} \end{bmatrix}. \qquad (16.12)$$

Definición de la Matriz

```
Iso = ( 1    1    1    1
        x1   x2   x3   x4
        y1   y2   y3   y4
        ux1  ux2  ux3  ux4
        uy1  uy2  uy3  uy4 );
```

Formulación Isoparamétrica

```
(Uno
 x
 y
 ux
 uy) = Iso.Nf
```

$$\left\{\left\{\frac{1}{4}(1-\eta)(1-\xi)+\frac{1}{4}(1+\eta)(1-\xi)+\frac{1}{4}(1-\eta)(1+\xi)+\frac{1}{4}(1+\eta)(1+\xi)\right\},\right.$$
$$\left\{\frac{1}{4}\,x1\,(1-\eta)(1-\xi)+\frac{1}{4}\,x4\,(1+\eta)(1-\xi)+\frac{1}{4}\,x2\,(1-\eta)(1+\xi)+\frac{1}{4}\,x3\,(1+\eta)(1+\xi)\right\},$$
$$\left\{\frac{1}{4}\,y1\,(1-\eta)(1-\xi)+\frac{1}{4}\,y4\,(1+\eta)(1-\xi)+\frac{1}{4}\,y2\,(1-\eta)(1+\xi)+\frac{1}{4}\,y3\,(1+\eta)(1+\xi)\right\},$$
$$\left\{\frac{1}{4}\,ux1\,(1-\eta)(1-\xi)+\frac{1}{4}\,ux4\,(1+\eta)(1-\xi)+\frac{1}{4}\,ux2\,(1-\eta)(1+\xi)+\frac{1}{4}\,ux3\,(1+\eta)(1+\xi)\right\},$$
$$\left.\left\{\frac{1}{4}\,uy1\,(1-\eta)(1-\xi)+\frac{1}{4}\,uy4\,(1+\eta)(1-\xi)+\frac{1}{4}\,uy2\,(1-\eta)(1+\xi)+\frac{1}{4}\,uy3\,(1+\eta)(1+\xi)\right\}\right\}$$

Condición Funciones de Forma

```
Uno
```

$$\frac{1}{4}(1-\eta)(1-\xi)+\frac{1}{4}(1+\eta)(1-\xi)+\frac{1}{4}(1-\eta)(1+\xi)+\frac{1}{4}(1+\eta)(1+\xi)$$

Interpolación de la Geometría

```
x
```

$$\frac{1}{4}\,x1\,(1-\eta)(1-\xi)+\frac{1}{4}\,x4\,(1+\eta)(1-\xi)+\frac{1}{4}\,x2\,(1-\eta)(1+\xi)+\frac{1}{4}\,x3\,(1+\eta)(1+\xi)$$

```
y
```

$$\frac{1}{4}\,y1\,(1-\eta)(1-\xi)+\frac{1}{4}\,y4\,(1+\eta)(1-\xi)+\frac{1}{4}\,y2\,(1-\eta)(1+\xi)+\frac{1}{4}\,y3\,(1+\eta)(1+\xi)$$

Interpolación de los Desplazamientos

```
ux
```

$$\frac{1}{4}\,ux1\,(1-\eta)(1-\xi)+\frac{1}{4}\,ux4\,(1+\eta)(1-\xi)+\frac{1}{4}\,ux2\,(1-\eta)(1+\xi)+\frac{1}{4}\,ux3\,(1+\eta)(1+\xi)$$

```
uy
```

$$\frac{1}{4}\,uy1\,(1-\eta)(1-\xi)+\frac{1}{4}\,uy4\,(1+\eta)(1-\xi)+\frac{1}{4}\,uy2\,(1-\eta)(1+\xi)+\frac{1}{4}\,uy3\,(1+\eta)(1+\xi)$$

Defnición General de la Interpolación de los Desplazamientos mediante Funciones de Forma

In[44]:=

```
InterpolaciónFN = Import["012.jpg"];
Show[InterpolaciónFN, ImageSize → 750]
```

Out[45]=

$$\mathbf{u}(x,y)=\begin{bmatrix}u_x(x,y)\\u_y(x,y)\end{bmatrix}=\begin{bmatrix}N_1^{(e)} & 0 & N_2^{(e)} & 0 & \dots & N_n^{(e)} & 0\\ 0 & N_1^{(e)} & 0 & N_2^{(e)} & \dots & 0 & N_n^{(e)}\end{bmatrix}\mathbf{u}^{(e)}=\mathbf{N}^{(e)}\mathbf{u}^{(e)}. \qquad (14.17)$$

Obtencion Desplazamientos en un Punto.

***** Matriz Funciones de Forma - N

```
Ne = (N1  0  N2  0  N3  0  N4  0
      0  N1  0  N2  0  N3  0  N4);
```

```
Ne // MatrixForm
```

$$\begin{pmatrix}\frac{1}{4}(1-\eta)(1-\xi) & 0 & \frac{1}{4}(1-\eta)(1+\xi) & 0 & \frac{1}{4}(1+\eta)(1+\xi) & 0 & \frac{1}{4}(1+\eta)(1-\xi \\ 0 & \frac{1}{4}(1-\eta)(1-\xi) & 0 & \frac{1}{4}(1-\eta)(1+\xi) & 0 & \frac{1}{4}(1+\eta)(1+\xi) & 0\end{pmatrix}$$

Vector Desplazamientos Nodales - DATOS

In[46]:=
```
DesplazamientosNodales = Import["013.jpg"];
Show[DesplazamientosNodales, ImageSize → 750]
```

Out[47]=

$$\mathbf{u}^{(e)} = [u_{x1} \quad u_{y1} \quad u_{x2} \quad \ldots \quad u_{xn} \quad u_{yn}]^T . \qquad (14.15)$$

```
ue = (ux1, uy1, ux2, uy2, ux3, uy3, ux4, uy4)ᵀ;
```

Vector Desplazamientos en un Punto en función de los nodales - COORDENADAS NATURALES

```
u[ξ_, η_] = Ne.ue;
u[ξ, η] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{4}\,ux1\,(1-\eta)\,(1-\xi) + \frac{1}{4}\,ux4\,(1+\eta)\,(1-\xi) + \frac{1}{4}\,ux2\,(1-\eta)\,(1+\xi) + \frac{1}{4}\,ux3\,(1+\eta)\,(1+\xi) \\ \frac{1}{4}\,uy1\,(1-\eta)\,(1-\xi) + \frac{1}{4}\,uy4\,(1+\eta)\,(1-\xi) + \frac{1}{4}\,uy2\,(1-\eta)\,(1+\xi) + \frac{1}{4}\,uy3\,(1+\eta)\,(1+\xi) \end{pmatrix}$$

No podemos obtener la función que nos proporcione los desplazamientos en función de x e y, u[x,y], por no ser posible obtener las coordenadas x e y conocidas las ξ y η

Obtención de las Deformaciones - 2a Forma: Desplazamientos Nodales

Relaciones Básicas - Problema Tensión Plana

*** Las deformaciones en el punto se calculan mediante esta relación vectorial en derivadas parciales

In[48]:=
```
TensiónPlanaM = Import["014.jpg"];
Show[TensiónPlanaM, ImageSize → 750]
```

Out[49]=

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix},$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}, \qquad (14.5)$$

$$\begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In[50]:=
```
TensiónPlanaC = Import["015.jpg"];
Show[TensiónPlanaC, ImageSize → 750]
```

Out[51]=

$$\mathbf{e} = \mathbf{D}\mathbf{u}, \qquad \sigma = \mathbf{E}\mathbf{e}, \qquad \mathbf{D}^T\sigma + \mathbf{b} = \mathbf{0}, \qquad (14.6)$$

In[52]:=
```
Show[InterpolaciónFN, ImageSize → 750]
```

Out[52]=

$$\mathbf{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} N_1^{(e)} & 0 & N_2^{(e)} & 0 & \ldots & N_n^{(e)} & 0 \\ 0 & N_1^{(e)} & 0 & N_2^{(e)} & \ldots & 0 & N_n^{(e)} \end{bmatrix} \mathbf{u}^{(e)} = \mathbf{N}^{(e)}\mathbf{u}^{(e)}. \qquad (14.17)$$

In[53]:=
```
Deformaciones = Import["016.jpg"];
Show[Deformaciones, ImageSize → 750]
```

Out[54]=

$$\mathbf{e}(x,y) = \begin{bmatrix} \frac{\partial N_1^{(e)}}{\partial x} & 0 & \frac{\partial N_2^{(e)}}{\partial x} & 0 & \dots & \frac{\partial N_n^{(e)}}{\partial x} & 0 \\ 0 & \frac{\partial N_1^{(e)}}{\partial y} & 0 & \frac{\partial N_2^{(e)}}{\partial y} & \dots & 0 & \frac{\partial N_n^{(e)}}{\partial y} \\ \frac{\partial N_1^{(e)}}{\partial y} & \frac{\partial N_1^{(e)}}{\partial x} & \frac{\partial N_2^{(e)}}{\partial y} & \frac{\partial N_2^{(e)}}{\partial x} & \dots & \frac{\partial N_n^{(e)}}{\partial y} & \frac{\partial N_n^{(e)}}{\partial x} \end{bmatrix} \mathbf{u}^{(e)} = \mathbf{B}\,\mathbf{u}^{(e)}. \quad (14.18)$$

Calculo Derivadas Parciales

Imposible obtener de forma directa las derivadas de la funciones de forma respecto a las coordenadas cartesianas, al ser las funciones de forma función de las coordenadas naturales.

In[18]:=
```
Show[Import["0004.jpg"], ImageSize → 1000]
```

Out[18]=

§17.2. Partial Derivative Computation

Partial derivatives of shape functions with respect to the Cartesian coordinates x and y are required for the strain and stress calculations. Because shape functions are not directly functions of x and y but of the natural coordinates ξ and η, the determination of Cartesian partial derivatives is not trivial. The derivative calculation procedure is presented below for the case of an arbitrary isoparametric quadrilateral element with n nodes.

Matriz Jacobiana

In[19]:=
```
Show[Import["0005.jpg"], ImageSize → 1000]
```

Out[19]=

§17.2.1. The Jacobian

In quadrilateral element derivations we will need the Jacobian of two-dimensional transformations that connect the differentials of $\{x, y\}$ to those of $\{\xi, \eta\}$ and vice-versa. Using the chain rule:

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \mathbf{J}^T \begin{bmatrix} d\xi \\ d\eta \end{bmatrix}, \quad \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \mathbf{J}^{-T} \begin{bmatrix} dx \\ dy \end{bmatrix}. \quad (17.1)$$

Here $\mathbf{J}$ denotes the Jacobian matrix of (x, y) with respect to (ξ, η), whereas $\mathbf{J}^{-1}$ is the Jacobian matrix of (ξ, η) with respect to (x, y):

$$\mathbf{J} = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \quad \mathbf{J}^{-1} = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}, \quad (17.2)$$

where $J = |\mathbf{J}| = \det(\mathbf{J}) = J_{11}J_{22} - J_{12}J_{21}$. In FEM work $\mathbf{J}$ and $\mathbf{J}^{-1}$ are called simply the *Jacobian* and *inverse Jacobian*, respectively; the fact that it is a matrix being understood. The scalar symbol

In[20]:=

```
Show[Import["0006.jpg"], ImageSize → 1000]
```

Out[20]=

J is reserved for the determinant of $\mathbf{J}$. In one dimension $\mathbf{J}$ and J coalesce. Jacobians play a crucial role in differential geometry. For the general definition of Jacobian matrix of a differential transformation, see Appendix D.

Remark 17.1. Observe that the matrices relating the differentials in (17.1) are the *transposes* of what we call $\mathbf{J}$ and $\mathbf{J}^{-1}$. The reason is that coordinate differentials transform as contravariant quantities: $dx = (\partial x/\partial\xi)\,d\xi + (\partial x/\partial\eta)\,d\eta$, etc. But Jacobians are arranged as in (17.2) because of earlier use in covariant transformations: $\partial\phi/\partial x = (\partial\xi/\partial x)(\partial\phi/\partial\xi) + (\partial\eta/\partial x)(\partial\phi/\partial\eta)$, as in (17.5) below.

The reader is cautioned that notations vary among application areas. As quoted in Appendix D, one author puts it this way: "When one does matrix calculus, one quickly finds that there are two kinds of people in this world: those who think the gradient is a row vector, and those who think it is a column vector."

Remark 17.2. To show that $\mathbf{J}$ and $\mathbf{J}^{-1}$ are in fact inverses of each other we form their product:

$$\mathbf{J}^{-1}\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial x}{\partial\eta}\frac{\partial\eta}{\partial x} & \frac{\partial y}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial y}{\partial\eta}\frac{\partial\eta}{\partial x} \\ \frac{\partial x}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial x}{\partial\eta}\frac{\partial\eta}{\partial y} & \frac{\partial y}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial y}{\partial\eta}\frac{\partial\eta}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad (17.3)$$

where we have taken into account that $x = x(\xi,\eta)$, $y = y(\xi,\eta)$ and the fact that x and y are independent coordinates. This proof would collapse, however, if instead of $\{\xi,\eta\}$ we had the triangular coordinates $\{\zeta_1,\zeta_2,\zeta_3\}$ because rectangular matrices have no conventional inverses. This case requires special handling and is covered in Chapter 24.

*** Definición de la Matriz Jacobiana

In[55]:=

```
MatrizJacobiana1 = Import["005.jpg"];
Show[MatrizJacobiana1, ImageSize → 750]
```

Out[56]=

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta} \\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\eta} \end{bmatrix} \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \mathbf{J}^T \begin{bmatrix} d\xi \\ d\eta \end{bmatrix}, \quad \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \begin{bmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial x} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \mathbf{J}^{-T} \begin{bmatrix} dx \\ dy \end{bmatrix}, \qquad (17.1)$$

In[57]:=

```
MatrizJacobiana2 = Import["006.jpg"];
Show[MatrizJacobiana2, ImageSize → 750]
```

Out[58]=

$$\mathbf{J} = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{bmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} \\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \qquad \mathbf{J}^{-1} = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\eta}{\partial x} \\ \frac{\partial\xi}{\partial y} & \frac{\partial\eta}{\partial y} \end{bmatrix} \qquad (17.2)$$

$$J = \begin{pmatrix} \partial_\xi x & \partial_\xi y \\ \partial_\eta x & \partial_\eta y \end{pmatrix};$$

```
J // MatrixForm
```

$$\begin{pmatrix} -\frac{1}{4}x1(1-\eta) + \frac{1}{4}x2(1-\eta) + \frac{1}{4}x3(1+\eta) - \frac{1}{4}x4(1+\eta) & -\frac{1}{4}y1(1-\eta) + \frac{1}{4}y2(1-\eta) + \frac{1}{4}y3(1+\eta) - \frac{1}{4}y4(1+\eta) \\ -\frac{1}{4}x1(1-\xi) + \frac{1}{4}x4(1-\xi) - \frac{1}{4}x2(1+\xi) + \frac{1}{4}x3(1+\xi) & -\frac{1}{4}y1(1-\xi) + \frac{1}{4}y4(1-\xi) - \frac{1}{4}y2(1+\xi) + \frac{1}{4}y3(1+\xi) \end{pmatrix}$$

▫ Jacobiano - Determinante de la Matriz Jacobiana

---> JACOBIANO - DETERMINANTE MATRIZ JACOBIANA

```
Jdet = Simplify[Det[J]]
```

$$\frac{1}{8}(x1\,y2 - x3\,y2 - x1\,y4 + x3\,y4 - x3\,y1\,\eta - x1\,y2\,\eta + x1\,y3\,\eta + x3\,y4\,\eta + x3\,y1\,\xi - x3\,y2\,\xi - x1\,y3\,\xi + x1\,y4\,\xi + x4\,(y1 - y3\,(1+\eta) - y1\,\xi + y2\,(\eta+\xi)) + x2\,(y1\,(-1+\eta) + y3\,(1+\xi) - y4\,(\eta+\xi)))$$

▫ Matriz Jacobiana Inversa

Obtención de la Matriz Jacobiana Inversa

In[59]:=
```
Show[MatrizJacobiana2, ImageSize → 750]
```

Out[59]=

$$\mathbf{J} = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \qquad \mathbf{J}^{-1} = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \quad (17.2)$$

```
(dξdx  dηdx
 dξdy  dηdy) = Simplify[Inverse[J]];
```

```
dξdy
```

```
(2 (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
   x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))
```

Derivadas Funciones Forma respecto Coordenadas Cartesianas

Obtención de las Derivadas de las Funciones de Forma mediante Regla de la Cadena

In[60]:=
```
DerivadasFuncionesForma = Import["007.jpg"];
Show[DerivadasFuncionesForma, ImageSize → 750]
```

Out[61]=

$$\frac{\partial N_i^{(e)}}{\partial x} = \frac{\partial N_i^{(e)}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_i^{(e)}}{\partial \eta}\frac{\partial \eta}{\partial x}, \qquad \frac{\partial N_i^{(e)}}{\partial y} = \frac{\partial N_i^{(e)}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_i^{(e)}}{\partial \eta}\frac{\partial \eta}{\partial y}. \quad (17.4)$$

```
dN1dx = ∂ξ N1 * dξdx + ∂η N1 * dηdx;
dN1dy = ∂ξ N1 * dξdy + ∂η N1 * dηdy;
```

```
dN2dx = ∂ξ N2 * dξdx + ∂η N2 * dηdx;
dN2dy = ∂ξ N2 * dξdy + ∂η N2 * dηdy;
```

```
dN3dx = ∂ξ N3 * dξdx + ∂η N3 * dηdx;
dN3dy = ∂ξ N3 * dξdy + ∂η N3 * dηdy;
```

```
dN4dx = ∂ξ N4 * dξdx + ∂η N4 * dηdx;
dN4dy = ∂ξ N4 * dξdy + ∂η N4 * dηdy;
```

---> DERIVADAS FUNCIONES DE FORMA RESPECTO COORDENADAS CARTESIANAS

```
dNx = Simplify[{dN1dx, dN2dx, dN3dx, dN4dx}];
```

```
dNy = Simplify[{dN1dy, dN2dy, dN3dy, dN4dy}];
```

```
dNx /. ncoor
```

$$\left\{\frac{\frac{1}{2}-\frac{\xi}{2}+\frac{1}{2}(-\eta+\xi)}{-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}}, \frac{\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}}{-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}}, \frac{\frac{1}{2}+\frac{\eta}{2}}{\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}}, \frac{1+\eta}{2\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)}\right\}$$

***** Matriz B - Be

```
Be = (dN1dx  0      dN2dx  0      dN3dx  0      dN4dx  0
      0      dN1dy  0      dN2dy  0      dN3dy  0      dN4dy
      dN1dy  dN1dx  dN2dy  dN2dx  dN3dy  dN3dx  dN4dy  dN4dx);
```

```
Be // MatrixForm
```

```
(  (y2+y1 (-1+η) -y2 η+ (y3-y4) (1+η)) (-1+ξ) / 2 (-x1 y2+x3 y2+x1 y4-x3 y4+x3 y1 η+x1 y2 η-x1 y3 η-x3 y4 η-x3 y1 ξ+x3 y2 ξ+x1 y3 ξ-x1 y4 ξ+x4 (y3 (1+η) +y1 (-1+ξ) -y2 (η+ξ)) +x2 (y1-y1 η-y3 (1+ξ) +y4 (η+ξ)))  -  / 2 (-:
   0
 - (x2+x1 (-1+η) -x2 η+ (x3-x4) (1+η)) (-1+ξ) / 2 (-x1 y2+x3 y2+x1 y4-x3 y4+x3 y1 η+x1 y2 η-x1 y3 η-x3 y4 η-x3 y1 ξ+x3 y2 ξ+x1 y3 ξ-x1 y4 ξ+x4 (y3 (1+η) +y1 (-1+ξ) -y2 (η+ξ)) +x2 (y1-y1 η-y3 (1+ξ) +y4 (η+ξ)))  +  / 2 (-
```

```
FullSimplify[dN1dx /. ncoor]
```

$$\frac{1}{2}\,(-1+\eta)$$

```
FullSimplify[Be /. ncoor] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2}(-1+\eta) & 0 & \frac{1-\eta}{2} & 0 & \frac{1+\eta}{2} & 0 & \frac{1}{2}(-1-\eta) & 0 \\ 0 & -1+\xi & 0 & -1-\xi & 0 & 1+\xi & 0 & 1-\xi \\ -1+\xi & \frac{1}{2}(-1+\eta) & -1-\xi & \frac{1-\eta}{2} & 1+\xi & \frac{1+\eta}{2} & 1-\xi & \frac{1}{2}(-1-\eta) \end{pmatrix}$$

▫ Vector Deformaciones en un Punto P - COORDENADAS NATURALES

```
e2[ξ_, η_] = Be.ue;
```

```
e2[ξ, η] // MatrixForm
```

$$\left(\text{ux4}\left(-\frac{(x2+x1\,(-1+\eta)-x2\,\eta+(x3-x4)\,(1+\eta))\,(1-\xi)}{2\,(-x1\,y2+x3\,y2+x1\,y4-x3\,y4+x3\,y1\,\eta+x1\,y2\,\eta-x1\,y3\,\eta-x3\,y4\,\eta-x3\,y1\,\xi+x3\,y2\,\xi+x1\,y3\,\xi-x1\,y4\,\xi+x4\,(y3\,(1+\eta)+y1\,(-1+\xi)-y2\,(\eta+\xi))+x2\,(y1-y1\,\eta-y3\,(1+\xi)+y4\,(\eta+\xi)))} \right.\right.$$

```
Dimensions[e2[ξ, η]]
```

{3, 1}

-:) tres filas por una columna

▫ Valor de la Deformación en x - exx

```
e2[ξ, η][[1, 1]]
```

```
ux4 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (1 - ξ)) /
      (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) -
    ((-1 - η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
      (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
 ux2 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (-1 - ξ)) /
      (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) -
    ((1 - η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
      (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
 ux1 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (-1 + ξ)) /
      (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) -
    ((-1 + η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
      (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
 ux3 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (1 + ξ)) /
      (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) -
    ((1 + η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
      (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))))
```

- Valor de la Deformación en y - eyy

```
e2[ξ, η][[2, 1]]
```

```
uy4 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (1 - ξ)) /
      (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) +
    ((-1 - η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
     (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
 uy2 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (-1 - ξ)) /
      (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) +
    ((1 - η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
     (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
 uy1 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (-1 + ξ)) /
      (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) +
    ((-1 + η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
     (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
 uy3 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (1 + ξ)) /
      (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) +
    ((1 + η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
     (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
          x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))))
```

Valor de la Deformación Tangencial xy - exy

```
e2[ξ, η][[3, 1]] / 2
```

```
1/2 (ux4
   (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (1 - ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η -
        x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
        x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) + ((-1 - η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
     (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
        x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
  ux2 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (-1 - ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
        x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
        x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) + ((1 - η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
     (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
        x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
  ux1 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (-1 + ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
        x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
        x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) + ((-1 + η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
     (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
        x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
  ux3 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (1 + ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
        x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
        x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) + ((1 + η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
     (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
        x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
  uy4 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (1 - ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
        x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
        x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) - ((-1 - η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
     (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
        x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
  uy2 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (-1 - ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
        x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
        x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) - ((1 - η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
     (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
        x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
  uy1 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (-1 + ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
        x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
        x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) - ((-1 + η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
     (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
        x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
  uy3 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (1 + ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
        x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
        x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) - ((1 + η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
     (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
        x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))))
```

Obtención de las Tensiones - Tensión Plana

Relaciones Básicas - Problema Tensión Plana

In[31]:=

```
Tensiones = Import["017.jpg"];
Show[Tensiones, ImageSize → 750]
```

Out[32]=

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{Ee}, \qquad (15.19)$$

In[29]:=
```
TensionesTP = Import["018.jpg"];
Show[TensionesTP, ImageSize → 750]
```

Out[30]=

[A:25] Suppose that the structural material is isotropic, with elastic modulus E and Poisson's ratio ν. The in-plane stress-strain relations for plane stress ($\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$) and plane strain ($e_{zz} = e_{xz} = e_{yz} = 0$) as given in any textbook on elasticity, are

$$\text{plane stress:} \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix},$$

$$\text{plane strain:} \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}. \qquad \text{(E14.1)}$$

***** Matriz del Material E - Em

$$Ee = \frac{Em}{1-\nu^2} * \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix};$$

Vector Tensiones en un Punto P

```
σ[ξ_, η_] = Ee.e2[ξ, η];
```

```
σ[ξ, η] // MatrixForm
```

```
( Em ν (uy4 (- (x2+x1 (-1+η) -x2 η+ (x3-x4) (1+η)) (1-ξ) / 2 (-x1 y2+x3 y2+x1 y4-x3 y4+x3 y1 η+x1 y2 η-x1 y3 η-x3 y4 η-x3 y1 ξ+x3 y2 ξ+x1 y3 ξ-x1 y4 ξ+x4 (y3 (1+η) +y1 (-1+ξ) -y2 (η+ξ)) +x2 (y1-y1 η-y3 (1+ξ) +y4 (η+ξ))) + ( / 2 (-x1 y2+x3 y2+x1 y4-x3 y4+x3 y1 η+x1 y2 η-x1 y3 η-x3 y4 ...
  Em (uy4 (- (x2+x1 (-1+η) -x2 η+ (x3-x4) (1+η)) (1-ξ) / 2 (-x1 y2+x3 y2+x1 y4-x3 y4+x3 y1 η+x1 y2 η-x1 y3 η-x3 y4 η-x3 y1 ξ+x3 y2 ξ+x1 y3 ξ-x1 y4 ξ+x4 (y3 (1+η) +y1 (-1+ξ) -y2 (η+ξ)) +x2 (y1-y1 η-y3 (1+ξ) +y4 (η+ξ))) + (-1 / 2 (-x1 y2+x3 y2+x1 y4-x3 y4+x3 y1 η+x1 y2 η-x1 y3 η-x3 y4 η-x ...
  Em (1-ν) (ux4 (- (x2+x1 (-1+η) -x2 η+ (x3-x4) (1+η)) (1-ξ) / 2 (-x1 y2+x3 y2+x1 y4-x3 y4+x3 y1 η+x1 y2 η-x1 y3 η-x3 y4 η-x3 y1 ξ+x3 y2 ξ+x1 y3 ξ-x1 y4 ξ+x4 (y3 (1+η) +y1 (-1+ξ) -y2 (η+ξ)) +x2 (y1-y1 η-y3 (1+ξ) +y4 (η+ξ))) + / 2 (-x1 y2+x3 y2+x1 y4-x3 y4+x3 y1 η+x1 y2 η-x1 y3 ...
```

```
Dimensions[σ[ξ, η]]
```

```
{3, 1}
```

-:) tres filas por una columna

▫ Valor de la Tensión en x - σxx

```
σ[ξ, η][[1, 1]]
```

```
   1
──────
1 - ν^2
 Em ν (uy4 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (1 - ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η -
              x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
              x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) + ((-1 - η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
         (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
             x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
    uy2 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (-1 - ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
              x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
              x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) + ((1 - η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
         (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
             x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
    uy1 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (-1 + ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
              x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
              x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) + ((-1 + η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
         (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
             x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
    uy3 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (1 + ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
              x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
              x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) + ((1 + η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
         (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
             x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))))) +
    1
 ──────  Em (ux4 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (1 - ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
 1 - ν^2
             x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
             x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) - ((-1 - η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
         (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
             x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
    ux2 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (-1 - ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
             x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
             x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) - ((1 - η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
         (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
             x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
    ux1 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (-1 + ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
             x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
             x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) - ((-1 + η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
         (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
             x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
    ux3 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (1 + ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
             x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
             x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) - ((1 + η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
         (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
             x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))))
```

▫ Valor de la Tensión en y - σyy

```
σ[ξ, η][[2, 1]]
```

$$
\begin{aligned}
&\frac{1}{1-\nu^2} \\
&Em\,(uy4\,(-((x2 + x1\,(-1 + \eta) - x2\,\eta + (x3 - x4)\,(1 + \eta))\,(1 - \xi))\,/\,(2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + x1\,y2\,\eta - x1 \\
&\quad y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + \\
&\quad x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi)))) + ((-1 - \eta)\,(x3 + x4 + x1\,(-1 + \xi) + x3\,\xi - x4\,\xi - x2\,(1 + \xi)))\,/ \\
&\quad (2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - \\
&\quad x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi))))) + \\
&uy2\,(-((x2 + x1\,(-1 + \eta) - x2\,\eta + (x3 - x4)\,(1 + \eta))\,(-1 - \xi))\,/\,(2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + \\
&\quad x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + \\
&\quad x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi)))) + ((1 - \eta)\,(x3 + x4 + x1\,(-1 + \xi) + x3\,\xi - x4\,\xi - x2\,(1 + \xi)))\,/ \\
&\quad (2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - \\
&\quad x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi))))) + \\
&uy1\,(-((x2 + x1\,(-1 + \eta) - x2\,\eta + (x3 - x4)\,(1 + \eta))\,(-1 + \xi))\,/\,(2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + \\
&\quad x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + \\
&\quad x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi)))) + ((-1 + \eta)\,(x3 + x4 + x1\,(-1 + \xi) + x3\,\xi - x4\,\xi - x2\,(1 + \xi)))\,/ \\
&\quad (2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - \\
&\quad x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi))))) + \\
&uy3\,(-((x2 + x1\,(-1 + \eta) - x2\,\eta + (x3 - x4)\,(1 + \eta))\,(1 + \xi))\,/\,(2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + \\
&\quad x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + \\
&\quad x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi)))) + ((1 + \eta)\,(x3 + x4 + x1\,(-1 + \xi) + x3\,\xi - x4\,\xi - x2\,(1 + \xi)))\,/ \\
&\quad (2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - \\
&\quad x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi)))))) + \\
&\frac{1}{1-\nu^2}\,Em\,\nu\,(ux4\,(((y2 + y1\,(-1 + \eta) - y2\,\eta + (y3 - y4)\,(1 + \eta))\,(1 - \xi))\,/ \\
&\quad (2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - \\
&\quad x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi)))) - \\
&\quad ((-1 - \eta)\,(y3 + y4 + y1\,(-1 + \xi) + y3\,\xi - y4\,\xi - y2\,(1 + \xi)))\,/ \\
&\quad (2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - \\
&\quad x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi))))) + \\
&ux2\,(((y2 + y1\,(-1 + \eta) - y2\,\eta + (y3 - y4)\,(1 + \eta))\,(-1 - \xi))\,/\,(2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + \\
&\quad x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + \\
&\quad x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi)))) - ((1 - \eta)\,(y3 + y4 + y1\,(-1 + \xi) + y3\,\xi - y4\,\xi - y2\,(1 + \xi)))\,/ \\
&\quad (2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - \\
&\quad x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi))))) + \\
&ux1\,(((y2 + y1\,(-1 + \eta) - y2\,\eta + (y3 - y4)\,(1 + \eta))\,(-1 + \xi))\,/\,(2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + \\
&\quad x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + \\
&\quad x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi)))) - ((-1 + \eta)\,(y3 + y4 + y1\,(-1 + \xi) + y3\,\xi - y4\,\xi - y2\,(1 + \xi)))\,/ \\
&\quad (2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - \\
&\quad x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi))))) + \\
&ux3\,(((y2 + y1\,(-1 + \eta) - y2\,\eta + (y3 - y4)\,(1 + \eta))\,(1 + \xi))\,/\,(2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + \\
&\quad x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + \\
&\quad x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi)))) - ((1 + \eta)\,(y3 + y4 + y1\,(-1 + \xi) + y3\,\xi - y4\,\xi - y2\,(1 + \xi)))\,/ \\
&\quad (2\,(-x1\,y2 + x3\,y2 + x1\,y4 - x3\,y4 + x3\,y1\,\eta + x1\,y2\,\eta - x1\,y3\,\eta - x3\,y4\,\eta - x3\,y1\,\xi + x3\,y2\,\xi + x1\,y3\,\xi - \\
&\quad x1\,y4\,\xi + x4\,(y3\,(1 + \eta) + y1\,(-1 + \xi) - y2\,(\eta + \xi)) + x2\,(y1 - y1\,\eta - y3\,(1 + \xi) + y4\,(\eta + \xi))))))
\end{aligned}
$$

▫ Valor de la Tensión Tangencial xy - σxy

```
σ[ξ, η][[3, 1]] / 2
```

$$\frac{1}{4\left(1-\nu^2\right)}$$

```
Em (1 - ν) (ux4 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (1 - ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
            x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
            x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) + ((-1 - η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
        (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
            x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
    ux2 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (-1 - ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
            x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
            x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) + ((1 - η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
        (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
            x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
    ux1 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (-1 + ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
            x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
            x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) + ((-1 + η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
        (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
            x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
    ux3 (-((x2 + x1 (-1 + η) - x2 η + (x3 - x4) (1 + η)) (1 + ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
            x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
            x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) + ((1 + η) (x3 + x4 + x1 (-1 + ξ) + x3 ξ - x4 ξ - x2 (1 + ξ))) /
        (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
            x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
    uy4 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (1 - ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
            x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
            x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) - ((-1 - η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
        (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
            x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
    uy2 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (-1 - ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
            x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
            x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) - ((1 - η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
        (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
            x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
    uy1 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (-1 + ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
            x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
            x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) - ((-1 + η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
        (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
            x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))) +
    uy3 (((y2 + y1 (-1 + η) - y2 η + (y3 - y4) (1 + η)) (1 + ξ)) / (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η +
            x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ - x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) +
            x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ)))) - ((1 + η) (y3 + y4 + y1 (-1 + ξ) + y3 ξ - y4 ξ - y2 (1 + ξ))) /
        (2 (-x1 y2 + x3 y2 + x1 y4 - x3 y4 + x3 y1 η + x1 y2 η - x1 y3 η - x3 y4 η - x3 y1 ξ + x3 y2 ξ + x1 y3 ξ -
            x1 y4 ξ + x4 (y3 (1 + η) + y1 (-1 + ξ) - y2 (η + ξ)) + x2 (y1 - y1 η - y3 (1 + ξ) + y4 (η + ξ))))))
```

▪ Matriz de Rigidez - Integrando

▫ Referencia

In[62]:=

```
MatrizRigidez = Import["008.jpg"];
Show[MatrizRigidez, ImageSize → 750]
```

Out[63]=

$$\mathbf{K}^{(e)} = \int_{\Omega^{(e)}} h\, \mathbf{B}^T \mathbf{E} \mathbf{B}\, d\Omega^{(e)}, \qquad (14.23)$$

In[33]:=

```
Show[Import["0008.jpg"], ImageSize → 1000]
```

Out[33]=

§17.4. The Stiffness Matrix

The stiffness matrix of a general plane stress element is given by the expression (14.23), which is reproduced here:

$$\mathbf{K}^e = \int_{\Omega^e} h\,\mathbf{B}^T\mathbf{E}\mathbf{B}\,d\Omega^e \qquad (17.18)$$

Of the terms that appear in (17.18) the strain-displacement matrix **B** has been discussed previously. The thickness h, if variable, may be interpolated via the shape functions. The stress-strain matrix **E** is usually constant in elastic problems, but we could in principle interpolate it as appropriate should it vary over the element. To integrate (17.18) numerically by a two-dimensional product Gauss rule, we have to reduce it to the canonical form (17.14), that is

$$\mathbf{K}^e = \int_{-1}^{1}\int_{-1}^{1} \mathbf{F}(\xi,\eta)\,d\xi\,d\eta. \qquad (17.19)$$

If ξ and η are the quadrilateral coordinates, everything in (17.19) already fits this form, except the element of area $d\Omega^e$.

In[34]:=

```
Show[Import["0009.jpg"], ImageSize → 1000]
```

Out[34]=

To complete the reduction we need to express $d\Omega^e$ in terms of the differentials $d\xi$ and $d\eta$. The desired relation is (see Remark below)

$$d\Omega^e = dx\,dy = \det\mathbf{J}\;d\xi\,d\eta = J\,d\xi\,d\eta. \qquad (17.20)$$

We therefore have

$$\mathbf{F}(\xi,\eta) = h\,\mathbf{B}^T\mathbf{E}\mathbf{B}\,\det\mathbf{J}. \qquad (17.21)$$

This matrix function can be numerically integrated over the domain $-1 \le \xi \le +1$, $-1 \le \eta \le +1$ by an appropriate Gauss product rule.

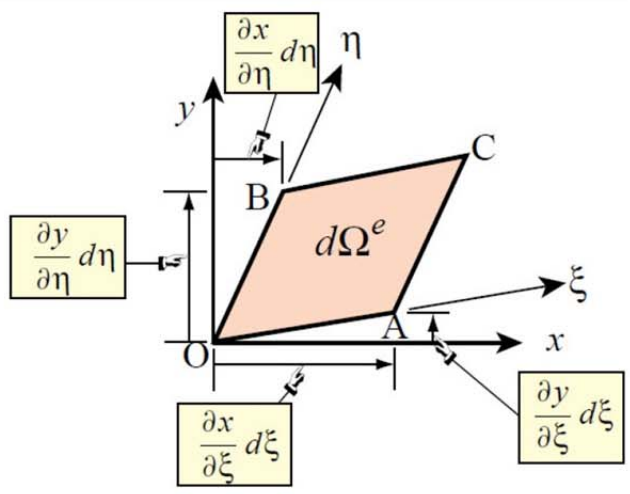


FIGURE 17.6. Geometric interpretation of the Jacobian-determinant formula.

In[35]:=

```
Show[Import["0010.jpg"], ImageSize → 1000]
```

Out[35]=

Remark 17.6. To geometrically justify the area transformation formula (17.20), consider the element of area OACB depicted in Figure 17.6. The area of this differential parallelogram can be computed as

$$dA = \vec{O}B \times \vec{O}A = \frac{\partial x}{\partial \xi}d\xi\,\frac{\partial y}{\partial \eta}d\eta - \frac{\partial x}{\partial \eta}d\eta\,\frac{\partial y}{\partial \xi}d\xi$$
$$= \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} d\xi\,d\eta = |\mathbf{J}|\,d\xi\,d\eta = \det\mathbf{J}\,d\xi\,d\eta. \qquad (17.22)$$

This formula can be extended to any number of dimensions, as shown in textbooks on differential geometry; for example [94,116,228].

Definición del Integrando, pues para calcular la integral hay que utilizar INTEGRACION NUMERICA

Definición del Integrando

```
IntegrandoKe = h * Transpose[Be].Ee.Be * Jdet;
```

```
IntegrandoKe // MatrixForm
```

$\frac{1}{8}$ h (x1 y2 − x3 y2 − x1 y4 + x3 y4 − x3 y1 η − x1 y2 η + x1 y3 η + x3 y4 η + x3 y1 ξ − x3 y2 ξ − x1 y3 ξ + x1 y4 ξ + x4 (y1 − y3 (1 + η)

$\frac{1}{8}$ h (x1 y2 − x3 y2 − x1 y4 + x3 y4 − x3 y1 η − x1 y2 η + x1 y3 η + x3 y4 η + x3 y1 ξ − x3 y2 ξ − x1 y3 ξ + x1 y4 ξ + x4 (y1 − y3 (1 + η

$\frac{1}{8}$ h (x1 y2 − x3 y2 − x1 y4 + x3 y4 − x3 y1 η − x1 y2 η + x1 y3 η + x3 y4 η + x3 y1 ξ − x3 y2 ξ − x1 y3 ξ + x1 y4 ξ + x4 (y1 − y3 (1 + η)

$\frac{1}{8}$ h (x1 y2 − x3 y2 − x1 y4 + x3 y4 − x3 y1 η − x1 y2 η + x1 y3 η + x3 y4 η + x3 y1 ξ − x3 y2 ξ − x1 y3 ξ + x1 y4 ξ + x4 (y1 − y3 (1 + η

$\frac{1}{8}$ h (x1 y2 − x3 y2 − x1 y4 + x3 y4 − x3 y1 η − x1 y2 η + x1 y3 η + x3 y4 η + x3 y1 ξ − x3 y2 ξ − x1 y3 ξ + x1 y4 ξ + x4 (y1 − y3 (1 + η)

$\frac{1}{8}$ h (x1 y2 − x3 y2 − x1 y4 + x3 y4 − x3 y1 η − x1 y2 η + x1 y3 η + x3 y4 η + x3 y1 ξ − x3 y2 ξ − x1 y3 ξ + x1 y4 ξ + x4 (y1 − y3 (1 + η

$\frac{1}{8}$ h (x1 y2 − x3 y2 − x1 y4 + x3 y4 − x3 y1 η − x1 y2 η + x1 y3 η + x3 y4 η + x3 y1 ξ − x3 y2 ξ − x1 y3 ξ + x1 y4 ξ + x4 (y1 − y3 (1 + η)

```
FullSimplify[IntegrandoKe /. ncoor /. mater] // MatrixForm
```

$\frac{9}{8}(7+3(-2+\eta)\eta+4(-2+\xi)\xi)$	$\frac{9}{2}(-1+\eta)(-1+\xi)$	$-\frac{9}{8}(-1+3(-2+\eta)\eta+4\xi^2)$	$-\frac{9}{2}(-1+$
$\frac{9}{2}(-1+\eta)(-1+\xi)$	$\frac{9}{8}(13+(-2+\eta)\eta+12(-2+\xi)\xi)$	$-\frac{9}{2}(-1+\eta)\xi$	$-\frac{9}{8}(-11+(-2+$
$-\frac{9}{8}(-1+3(-2+\eta)\eta+4\xi^2)$	$-\frac{9}{2}(-1+\eta)\xi$	$\frac{9}{8}(7+3(-2+\eta)\eta+4\xi(2+\xi))$	$\frac{9}{2}(-1+\eta)$
$-\frac{9}{2}(-1+\eta)\xi$	$-\frac{9}{8}(-11+(-2+\eta)\eta+12\xi^2)$	$\frac{9}{2}(-1+\eta)(1+\xi)$	$\frac{9}{8}(13+(-2+\eta)\eta$
$\frac{9}{8}(-7+3\eta^2+4\xi^2)$	$\frac{9}{2}(-1+\eta\xi)$	$-\frac{9}{8}(1+3\eta^2+4\xi(2+\xi))$	$-\frac{9}{2}\eta(1$
$\frac{9}{2}(-1+\eta\xi)$	$\frac{9}{8}(-13+\eta^2+12\xi^2)$	$-\frac{9}{2}\eta(1+\xi)$	$-\frac{9}{8}(11+\eta^2+12$
$-\frac{9}{8}(1+3\eta^2+4(-2+\xi)\xi)$	$-\frac{9}{2}\eta(-1+\xi)$	$\frac{9}{8}(-7+3\eta^2+4\xi^2)$	$\frac{9}{2}(1+\eta$
$-\frac{9}{2}\eta(-1+\xi)$	$-\frac{9}{8}(11+\eta^2+12(-2+\xi)\xi)$	$\frac{9}{2}(1+\eta\xi)$	$\frac{9}{8}(-13+\eta^2$

Integración Numérica

Número de Puntos de Gauss Mínimo paa Conseguir Suficiencia de Rango.

```
NF = NNodos * 2.;
```

$$NG = \frac{NF - 3}{3}$$

$$\frac{1}{3}(-3 + 2.\ NNodos)$$

Se necesitan como mínimo 2 Puntos -- Regla 2 x 2 minima

▫ Reglas de Cuadratura bidimensionales.

In[36]:=
```
Show[Import["0011.jpg"], ImageSize → 1000]
```

Out[36]=

§17.3.3. Two Dimensional Rules

The simplest two-dimensional Gauss rules are called *product rules*. They are obtained by applying the one-dimensional rules to each independent variable in turn. To apply these rules we must first reduce the integrand to the canonical form:

$$\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{1} d\eta \int_{-1}^{1} F(\xi,\eta)\,d\xi. \tag{17.14}$$

Once this is done we can process numerically each integral in turn:

$$\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{1} d\eta \int_{-1}^{1} F(\xi,\eta)\,d\xi \approx \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} w_i w_j F(\xi_i,\eta_j). \tag{17.15}$$

where p_1 and p_2 are the number of Gauss points in the ξ and η directions, respectively. Usually the same number $p = p_1 = p_2$ is chosen if the shape functions are taken to be the same in the ξ and η directions. This is in fact the case for all quadrilateral elements presented here. The first four two-dimensional Gauss product rules with $p = p_1 = p_2$ are illustrated in Figure 17.4.

In[37]:=
```
Show[Import["0012.jpg"], ImageSize → 1000]
```

Out[37]=

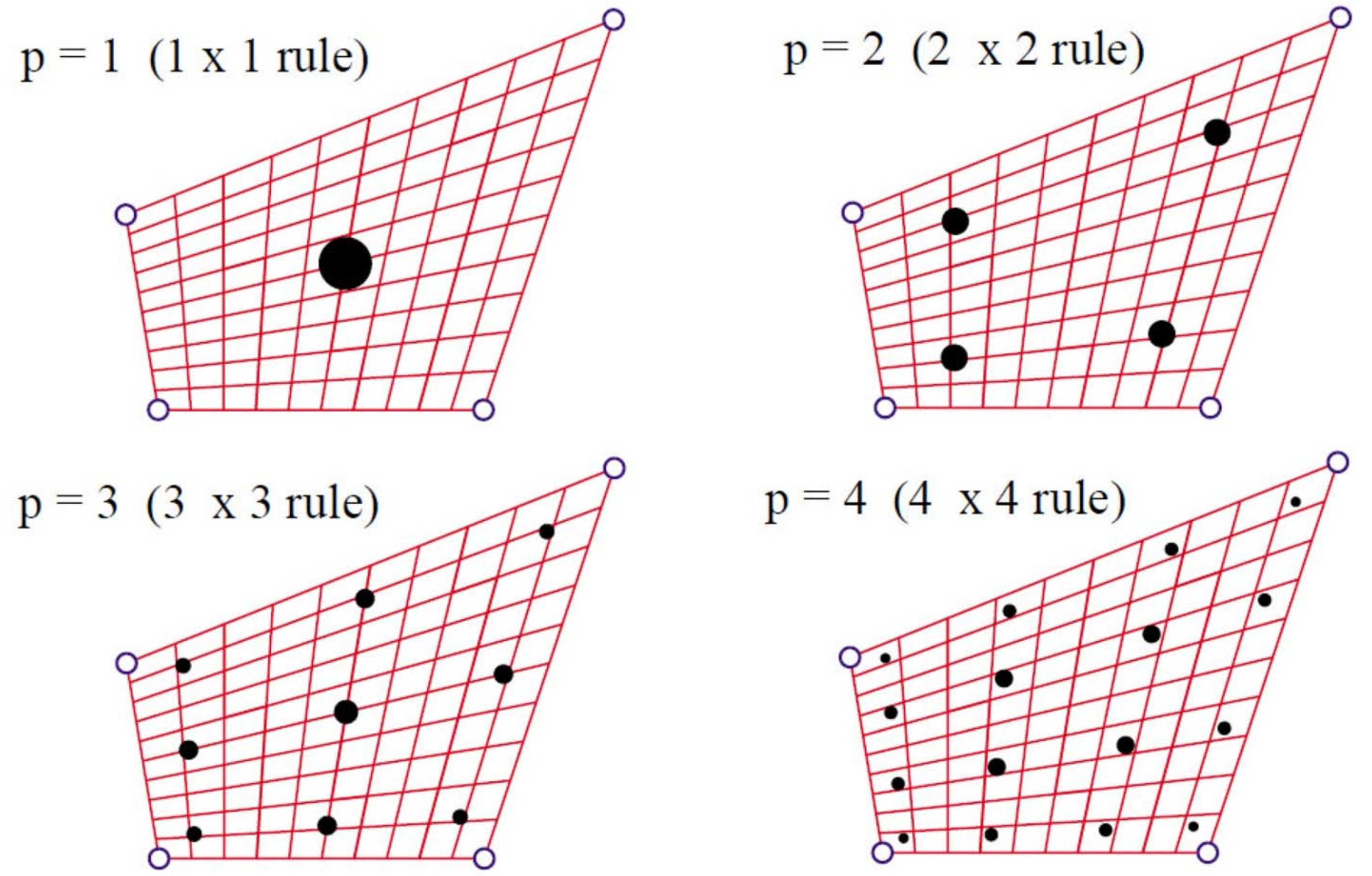


FIGURE 17.4. The first four two-dimensional Gauss product rules $p = 1, 2, 3, 4$ depicted over a straight-sided quadrilateral region. Sample points are marked with black circles. The areas of these circles are proportional to the integration weights.

▫ Opción 2: Definición de Carlos Felippa.

```
QuadGaussRuleInfo[{rule_, numer_}, point_] :=
  Module[{xi, eta, p1, p2, i1, i2, w1, w2, k, info = Null}, If[Length[rule] == 2, {p1, p2} = rule, p1 = p2 = rule];
   If[Length[point] == 2, {i1, i2} = point, k = point; i2 = Floor[(k - 1) / p1] + 1; i1 = k - p1 * (i2 - 1)];
   {xi, w1} = LineGaussRuleInfo[{p1, numer}, i1];
   {eta, w2} = LineGaussRuleInfo[{p2, numer}, i2];
   info = {{xi, eta}, w1 * w2};
   If[numer, Return[N[info]], Return[Simplify[info]]];];
```

```
LineGaussRuleInfo[{rule_, numer_}, point_] :=
  Module[{g2 = {-1, 1} / Sqrt[3], w3 = {5 / 9, 8 / 9, 5 / 9}, g3 = {-Sqrt[3 / 5], 0, Sqrt[3 / 5]},
    w4 = {(1 / 2) - Sqrt[5 / 6] / 6, (1 / 2) + Sqrt[5 / 6] / 6, (1 / 2) + Sqrt[5 / 6] / 6, (1 / 2) - Sqrt[5 / 6] / 6},
    g4 = {-Sqrt[(3 + 2 * Sqrt[6 / 5]) / 7], -Sqrt[(3 - 2 * Sqrt[6 / 5]) / 7],
      Sqrt[(3 - 2 * Sqrt[6 / 5]) / 7], Sqrt[(3 + 2 * Sqrt[6 / 5]) / 7]}, g5 = {-Sqrt[5 + 2 * Sqrt[10 / 7]],
       -Sqrt[5 - 2 * Sqrt[10 / 7]], 0, Sqrt[5 - 2 * Sqrt[10 / 7]], Sqrt[5 + 2 * Sqrt[10 / 7]]} / 3,
    w5 = {322 - 13 * Sqrt[70], 322 + 13 * Sqrt[70], 512, 322 + 13 * Sqrt[70], 322 - 13 * Sqrt[70]} / 900,
    i = point, p = rule, info = {Null, 0}},
   If[p == 1, info = {0, 2}];
   If[p == 2, info = {g2[[i]], 1}];
   If[p == 3, info = {g3[[i]], w3[[i]]}];
   If[p == 4, info = {g4[[i]], w4[[i]]}];
   If[p == 5, info = {g5[[i]], w5[[i]]}];
   If[numer, Return[N[info]], Return[Simplify[info]]];
  ];
```

▫ Regla 2 x 2 - Puntos y factiores ponderacion.

```
QuadGaussRuleInfo[{2, False}, 1]
```

$$\left\{\left\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\}, 1\right\}$$

```
QuadGaussRuleInfo[{2, False}, 2]
```

$$\left\{\left\{\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\}, 1\right\}$$

```
QuadGaussRuleInfo[{2, False}, 3]
```

$$\left\{\left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}, 1\right\}$$

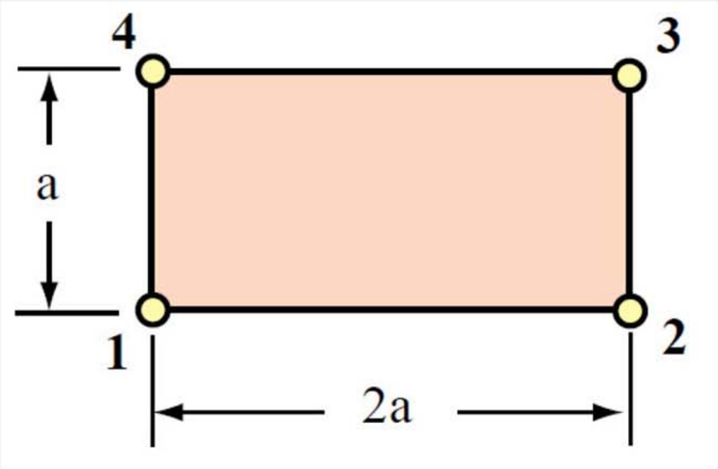
```
QuadGaussRuleInfo[{2, False}, 4]
```

$$\left\{\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}, 1\right\}$$

TEST DEL RECTANGULO.

Definición de la Geometría.

```
RectanguloTest = Import["RECTANGULO TEST.jpg"];
Show[RectanguloTest, ImageSize → 450]
```



Definición de los Nodos

```
ncoor = {x1 → 0, y1 → 0, x2 → 1, y2 → 0, x3 → 1, y3 → 1 / 2, x4 → 0, y4 → 1 / 2}
```

$\{x1 \to 0, y1 \to 0, x2 \to 1, y2 \to 0, x3 \to 1, y3 \to \frac{1}{2}, x4 \to 0, y4 \to \frac{1}{2}\}$

Definición del Material y del espesor.

```
Ee
```

$\left\{\left\{\frac{Em}{1-\nu^2}, \frac{Em\,\nu}{1-\nu^2}, 0\right\}, \left\{\frac{Em\,\nu}{1-\nu^2}, \frac{Em}{1-\nu^2}, 0\right\}, \left\{0, 0, \frac{Em\,(1-\nu)}{2\,(1-\nu^2)}\right\}\right\}$

```
mater = {Em → 96, ν → 1 / 3, h → 1}
```

$\{Em \to 96, \nu \to \frac{1}{3}, h \to 1\}$

```
Ee /. mater // MatrixForm
```

$$\begin{pmatrix} 108 & 36 & 0 \\ 36 & 108 & 0 \\ 0 & 0 & 36 \end{pmatrix}$$

Calculo de la Matriz de Rigidez.

La Matriz de Rigidez se obtiene calculando el integrando en cada punto de integracion de Gauss, multiplicando este valor por el coeficiente de ponderación correspondiente, y sumando para cada uno de los cuatro puntos de integracion considerados.

1er Punto.

Las coordenadas naturales del punto son las que se indican, y el factor de ponderación en esta regla es 1.

```
Datos1 = QuadGaussRuleInfo[{2, False}, 1]
```

$\left\{\left\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\}, 1\right\}$

```
Dimensions[Datos1]
```

{2}

```
Datos1[[1]]
```

$\left\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\}$

```
Dimensions[Datos1[[1]]]
```

{2}

```
Datos1[[1]][[1]]
```

$-\frac{1}{\sqrt{3}}$

```
Datos1[[1]][[2]]
```

$-\frac{1}{\sqrt{3}}$

```
Integrando1 = IntegrandoKe /. ncoor /. mater;
% // MatrixForm
```

$$\left(\begin{array}{ll}
\frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{27(-1+\eta)^2}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\frac{36(-1+\xi)^2}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}\right) & \frac{9(-1+\eta)(-1+\xi)\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)}{2\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2} \\
\frac{9(-1+\eta)(-1+\xi)\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)}{2\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2} & \frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{9(-1+\eta)^2}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\ldots\right. \\
\frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{27(1-\eta)(-1+\eta)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\frac{36(-1-\xi)(-1+\xi)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}\right) & \frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{18(-1+\eta)(-1-\xi)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\frac{18}{\ldots}\right. \\
\frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{18(-1+\eta)(-1-\xi)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\frac{18(1-\eta)(-1+\xi)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}\right) & \frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{9(1-\eta)(-1+\eta)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\frac{108}{\ldots}\right. \\
\frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{27(-1+\eta)(1+\eta)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\frac{36(-1+\xi)(1+\xi)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}\right) & \frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{18(1+\eta)(-1+\xi)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\frac{18}{\ldots}\right. \\
\frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{18(1+\eta)(-1+\xi)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\frac{18(-1+\eta)(1+\xi)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}\right) & \frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{9(-1+\eta)(1+\eta)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\frac{108}{\ldots}\right. \\
\frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{27(-1-\eta)(-1+\eta)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\frac{36(1-\xi)(-1+\xi)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}\right) & \frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{18(-1+\eta)(1-\xi)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\frac{18}{\ldots}\right. \\
\frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{18(-1+\eta)(1-\xi)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\frac{18(-1-\eta)(-1+\xi)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}\right) & \frac{1}{8}\left(\frac{1}{2}+\frac{\eta}{2}+\frac{1}{2}(-\eta-\xi)+\frac{1+\xi}{2}\right)\left(\frac{9(-1-\eta)(-1+\eta)}{\left(-\frac{1}{2}-\frac{\eta}{2}+\frac{1}{2}(-1-\xi)+\frac{\eta+\xi}{2}\right)^2}+\frac{108}{\ldots}\right.
\end{array}\right.$$

```
Integrando1 = FullSimplify[IntegrandoKe /. ncoor /. mater /. {ξ → Datos1[[1]][[1]], η -> Datos1[[1]][[2]]}];
% // MatrixForm
```

$$\left(\begin{array}{cccccccc}
\frac{21}{4}\left(2+\sqrt{3}\right) & 3\left(2+\sqrt{3}\right) & -\frac{3}{4}\left(2+3\sqrt{3}\right) & -\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{21}{4} & -3 & -\frac{3}{4}\left(5+4\sqrt{3}\right) & -\frac{3}{2}(1+\ldots \\
3\left(2+\sqrt{3}\right) & \frac{39}{4}\left(2+\sqrt{3}\right) & -\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{3}{4}\left(-10+\sqrt{3}\right) & -3 & -\frac{39}{4} & -\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{69}{4}-\ldots \\
-\frac{3}{4}\left(2+3\sqrt{3}\right) & -\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{3}{4}\left(-14+\sqrt{3}\right) & -3 & -\frac{15}{4}+3\sqrt{3} & \frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{21}{4} & 6 \\
-\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{3}{4}\left(-10+\sqrt{3}\right) & -3 & \frac{1}{4}\left(78-33\sqrt{3}\right) & \frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{69}{4}+9\sqrt{3} & 6 & -\frac{3}{4}\ldots \\
-\frac{21}{4} & -3 & -\frac{15}{4}+3\sqrt{3} & \frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{21}{4}\left(-2+\sqrt{3}\right) & 6-3\sqrt{3} & \frac{3}{4}\left(-2+3\sqrt{3}\right) & \frac{3}{2}(-1+\ldots \\
-3 & -\frac{39}{4} & \frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{69}{4}+9\sqrt{3} & 6-3\sqrt{3} & -\frac{39}{4}\left(-2+\sqrt{3}\right) & \frac{3}{2}\left(-1+\sqrt{3}\right) & \frac{3}{4}(10+\ldots \\
-\frac{3}{4}\left(5+4\sqrt{3}\right) & -\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{21}{4} & 6 & \frac{3}{4}\left(-2+3\sqrt{3}\right) & \frac{3}{2}\left(-1+\sqrt{3}\right) & \frac{3}{4}\left(14+\sqrt{3}\right) & -3\ldots \\
-\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{69}{4}-9\sqrt{3} & 6 & -\frac{39}{4} & \frac{3}{2}\left(-1+\sqrt{3}\right) & \frac{3}{4}\left(10+\sqrt{3}\right) & -3 & \frac{3}{4}(26+1\ldots
\end{array}\right.$$

▫ 2er Punto.

Las coordenadas naturales del punto son las que se indican, y el factor de ponderación en esta regla es 1.

```
Datos2 = QuadGaussRuleInfo[{2, False}, 2]
```

$\left\{\left\{\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\}, 1\right\}$

```
Integrando2 = FullSimplify[IntegrandoKe /. ncoor /. mater /. {ξ → Datos2[[1]][[1]], η -> Datos2[[1]][[2]]}];
% // MatrixForm
```

$$\begin{pmatrix}
-\frac{3}{4}\left(-14+\sqrt{3}\right) & 3 & -\frac{3}{4}\left(2+3\sqrt{3}\right) & \frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{21}{4} & -6 & -\frac{15}{4}+3\sqrt{3} & -\frac{3}{2}(-1 \\
3 & \frac{1}{4}\left(78-33\sqrt{3}\right) & \frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{3}{4}\left(-10+\sqrt{3}\right) & -6 & -\frac{39}{4} & -\frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{69}{4}+\ldots \\
-\frac{3}{4}\left(2+3\sqrt{3}\right) & \frac{3}{2}\left(1+\sqrt{3}\right) & \frac{21}{4}\left(2+\sqrt{3}\right) & -3\left(2+\sqrt{3}\right) & -\frac{3}{4}\left(5+4\sqrt{3}\right) & \frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{21}{4} & 3 \\
\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{3}{4}\left(-10+\sqrt{3}\right) & -3\left(2+\sqrt{3}\right) & \frac{39}{4}\left(2+\sqrt{3}\right) & \frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{69}{4}-9\sqrt{3} & 3 & -\frac{3}{4}\ldots \\
-\frac{21}{4} & -6 & -\frac{3}{4}\left(5+4\sqrt{3}\right) & \frac{3}{2}\left(1+\sqrt{3}\right) & \frac{3}{4}\left(14+\sqrt{3}\right) & 3 & \frac{3}{4}\left(-2+3\sqrt{3}\right) & -\frac{3}{2}(-1 \\
-6 & -\frac{39}{4} & \frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{69}{4}-9\sqrt{3} & 3 & \frac{3}{4}\left(26+11\sqrt{3}\right) & -\frac{3}{2}\left(-1+\sqrt{3}\right) & \frac{3}{4}(10+ \\
-\frac{15}{4}+3\sqrt{3} & -\frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{21}{4} & 3 & \frac{3}{4}\left(-2+3\sqrt{3}\right) & -\frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{21}{4}\left(-2+\sqrt{3}\right) & 3(-2+ \\
-\frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{69}{4}+9\sqrt{3} & 3 & -\frac{39}{4} & -\frac{3}{2}\left(-1+\sqrt{3}\right) & \frac{3}{4}\left(10+\sqrt{3}\right) & 3\left(-2+\sqrt{3}\right) & -\frac{39}{4}(-2
\end{pmatrix}$$

▫ 3er Punto.

Las coordenadas naturales del punto son las que se indican, y el factor de ponderación en esta regla es 1.

```
Datos3 = QuadGaussRuleInfo[{2, False}, 3]
```

$$\left\{\left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}, 1\right\}$$

```
Integrando3 = FullSimplify[IntegrandoKe /. ncoor /. mater /. {ξ → Datos3[[1]][[1]], η -> Datos3[[1]][[2]]}];
% // MatrixForm
```

$$\begin{pmatrix}
\frac{3}{4}\left(14+\sqrt{3}\right) & 3 & \frac{3}{4}\left(-2+3\sqrt{3}\right) & -\frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{21}{4} & -6 & -\frac{3}{4}\left(5+4\sqrt{3}\right) & \frac{3}{2}(1+ \\
3 & \frac{3}{4}\left(26+11\sqrt{3}\right) & -\frac{3}{2}\left(-1+\sqrt{3}\right) & \frac{3}{4}\left(10+\sqrt{3}\right) & -6 & -\frac{39}{4} & \frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{69}{4}-\ldots \\
\frac{3}{4}\left(-2+3\sqrt{3}\right) & -\frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{21}{4}\left(-2+\sqrt{3}\right) & 3\left(-2+\sqrt{3}\right) & -\frac{15}{4}+3\sqrt{3} & -\frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{21}{4} & 3 \\
-\frac{3}{2}\left(-1+\sqrt{3}\right) & \frac{3}{4}\left(10+\sqrt{3}\right) & 3\left(-2+\sqrt{3}\right) & -\frac{39}{4}\left(-2+\sqrt{3}\right) & -\frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{69}{4}+9\sqrt{3} & 3 & -\frac{3}{4}\ldots \\
-\frac{21}{4} & -6 & -\frac{15}{4}+3\sqrt{3} & -\frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{3}{4}\left(-14+\sqrt{3}\right) & 3 & -\frac{3}{4}\left(2+3\sqrt{3}\right) & \frac{3}{2}(1+ \\
-6 & -\frac{39}{4} & -\frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{69}{4}+9\sqrt{3} & 3 & \frac{1}{4}\left(78-33\sqrt{3}\right) & \frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{3}{4}(-10 \\
-\frac{3}{4}\left(5+4\sqrt{3}\right) & \frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{21}{4} & 3 & -\frac{3}{4}\left(2+3\sqrt{3}\right) & \frac{3}{2}\left(1+\sqrt{3}\right) & \frac{21}{4}\left(2+\sqrt{3}\right) & -3(2+ \\
\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{69}{4}-9\sqrt{3} & 3 & -\frac{39}{4} & \frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{3}{4}\left(-10+\sqrt{3}\right) & -3\left(2+\sqrt{3}\right) & \frac{39}{4}(2+
\end{pmatrix}$$

▫ 4er Punto.

Las coordenadas naturales del punto son las que se indican, y el factor de ponderación en esta regla es 1.

```
Datos4 = QuadGaussRuleInfo[{2, False}, 4]
```

$$\left\{\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}, 1\right\}$$

```
Integrando4 = FullSimplify[IntegrandoKe /. ncoor /. mater /. {ξ → Datos4[[1]][[1]], η -> Datos4[[1]][[2]]}];
% // MatrixForm
```

$$\begin{pmatrix}
-\frac{21}{4}\left(-2+\sqrt{3}\right) & 6-3\sqrt{3} & \frac{3}{4}\left(-2+3\sqrt{3}\right) & \frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{21}{4} & -3 & -\frac{15}{4}+3\sqrt{3} & \frac{3}{2}(-1+ \\
6-3\sqrt{3} & -\frac{39}{4}\left(-2+\sqrt{3}\right) & \frac{3}{2}\left(-1+\sqrt{3}\right) & \frac{3}{4}\left(10+\sqrt{3}\right) & -3 & -\frac{39}{4} & \frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{69}{4}+\ldots \\
\frac{3}{4}\left(-2+3\sqrt{3}\right) & \frac{3}{2}\left(-1+\sqrt{3}\right) & \frac{3}{4}\left(14+\sqrt{3}\right) & -3 & -\frac{3}{4}\left(5+4\sqrt{3}\right) & -\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{21}{4} & 6 \\
\frac{3}{2}\left(-1+\sqrt{3}\right) & \frac{3}{4}\left(10+\sqrt{3}\right) & -3 & \frac{3}{4}\left(26+11\sqrt{3}\right) & -\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{69}{4}-9\sqrt{3} & 6 & -\frac{3}{4}\ldots \\
-\frac{21}{4} & -3 & -\frac{3}{4}\left(5+4\sqrt{3}\right) & -\frac{3}{2}\left(1+\sqrt{3}\right) & \frac{21}{4}\left(2+\sqrt{3}\right) & 3\left(2+\sqrt{3}\right) & -\frac{3}{4}\left(2+3\sqrt{3}\right) & -\frac{3}{2}(1+ \\
-3 & -\frac{39}{4} & -\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{69}{4}-9\sqrt{3} & 3\left(2+\sqrt{3}\right) & \frac{39}{4}\left(2+\sqrt{3}\right) & -\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{3}{4}(-10 \\
-\frac{15}{4}+3\sqrt{3} & \frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{21}{4} & 6 & -\frac{3}{4}\left(2+3\sqrt{3}\right) & -\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{3}{4}\left(-14+\sqrt{3}\right) & -3\ldots \\
\frac{3}{2}\left(-1+\sqrt{3}\right) & -\frac{69}{4}+9\sqrt{3} & 6 & -\frac{39}{4} & -\frac{3}{2}\left(1+\sqrt{3}\right) & -\frac{3}{4}\left(-10+\sqrt{3}\right) & -3 & \frac{1}{4}(78-3\ldots
\end{pmatrix}$$

Matriz Rigidez

```
Ke = FullSimplify[Integrando1 + Integrando2 + Integrando3 + Integrando4];
% // MatrixForm
```

42	18	-6	0	-21	-18	-15	0
18	78	0	30	-18	-39	0	-69
-6	0	42	-18	-15	0	-21	18
0	30	-18	78	0	-69	18	-39
-21	-18	-15	0	42	18	-6	0
-18	-39	0	-69	18	78	0	30
-15	0	-21	18	-6	0	42	-18
0	-69	18	-39	0	30	-18	78

```
Chop[N[Ke]] // MatrixForm
```

42.	18.	-6.	0	-21.	-18.	-15.	0
18.	78.	0	30.	-18.	-39.	0	-69.
-6.	0	42.	-18.	-15.	0	-21.	18.
0	30.	-18.	78.	0	-69.	18.	-39.
-21.	-18.	-15.	0	42.	18.	-6.	0
-18.	-39.	0	-69.	18.	78.	0	30.
-15.	0	-21.	18.	-6.	0	42.	-18.
0	-69.	18.	-39.	0	30.	-18.	78.

Comprobación de la suficiencia de rango

La comprobación de la suficiencia de rango de la Matriz de Rigidez consiste en obtener los valores propios y comprobar que solo tres de ellos son nulos. Que se corresponden con los tres modos de cuerpo rigido existentes.

```
Chop[Eigenvalues[N[Ke]]]
```

{223.64, 90., 78., 46.3603, 42., 0, 0, 0}