UNIVERSIDAD POLITECNICA DE VALENCIA

DEPARTAMENTO DE INGENIERIA MECANICA Y DE MATERIALES

ELEMENTOS FINITOS

(E.T.S.I.I.V)

FORMULACION DE ELEMENTOS FINITOS

LECCION 5.- REPRESENTACION ISOPARAMETRICA

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001 DIFICULTADES PARA EXTENDER LA TECNICA PRESENTADA PARA ETL A OTROS ELEMENTOS CURSO 2004-5

1. The construction of shape functions that satisfy consistency requirements for higher order elements with curved boundaries becomes increasingly complicated.
2. Integrals that appear in the expressions of the element stiffness matrix and consistent nodal force vector can no longer be carried out in closed form.

001 CONCEPTOS FUNDAMENTALES QUE PERMITEN SUPERAR ESAS DIFICULTADES: CURSO 2004-5

* REPRESENTACION ISOPARAMETRICA
* CUADRATURA NUMERICA

These difficulties can be overcome through the concepts of *isoparametric elements* and *numerical quadrature*, respectively. The combination of these two ideas transformed the field of finite element methods in the late 1960s. Together they support a good portion of what is presently used in production finite element programs.

001 REPRESENTACION ISOPARAMETRICA DE EFs CURSO 2004-5

Geometry and displacements
are represented by
same set of shape functions (iso = same)

Advantages

Unification: same steps for all elements
No need to distinguish straight vs. curved side elements
Quick construction of shape functions

Before Isoparametric Concept was Discovered, FEM Developers Did "SuperParametric" Elems

Element shape functions refined,
more nodes and DOFs added

But element geometry was kept simple
with straight sides

001 ECUACIONES QUE DEFINEN EL ELEMENTO TRIANGULAR LINEAL CURSO 2004-5

For the 3 Node Triangle

Geometric Description

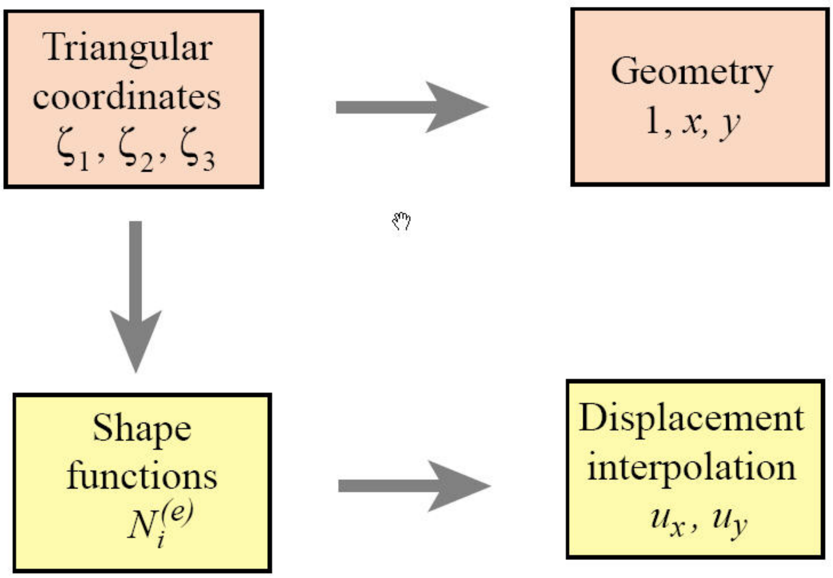
$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

Displacement Interpolation

$$u_x = u_{x1} N_1^{(e)} + u_{x2} N_2^{(e)} + u_{x3} N_3^{(e)} = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3$$
$$u_y = u_{y1} N_1^{(e)} + u_{y2} N_2^{(e)} + u_{y3} N_3^{(e)} = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3$$

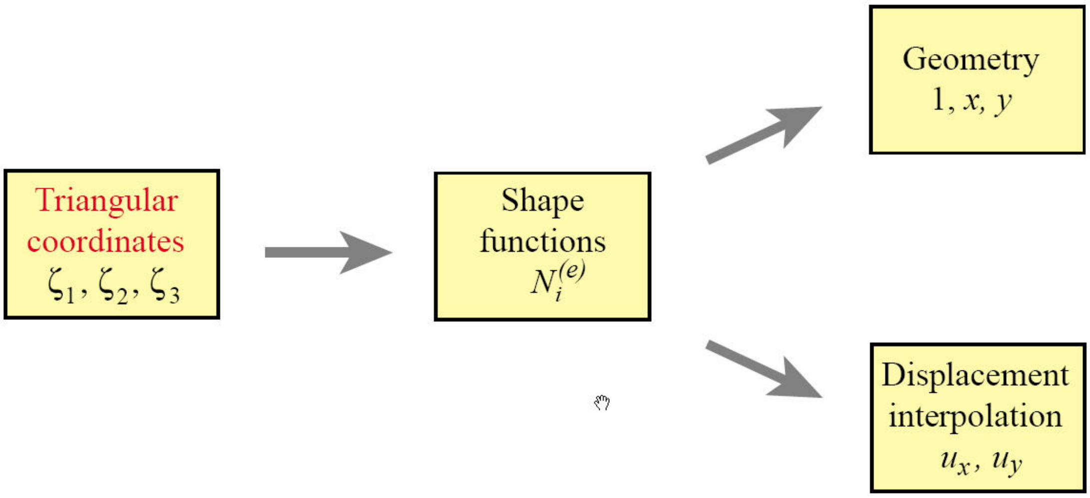
001 REPRESENTACION “SUPERPARAMETRICA” DEL ELEMENTO TRIANGULAR LINEAL CURSO 2004-5

SuperParametric Representation (Triangles)



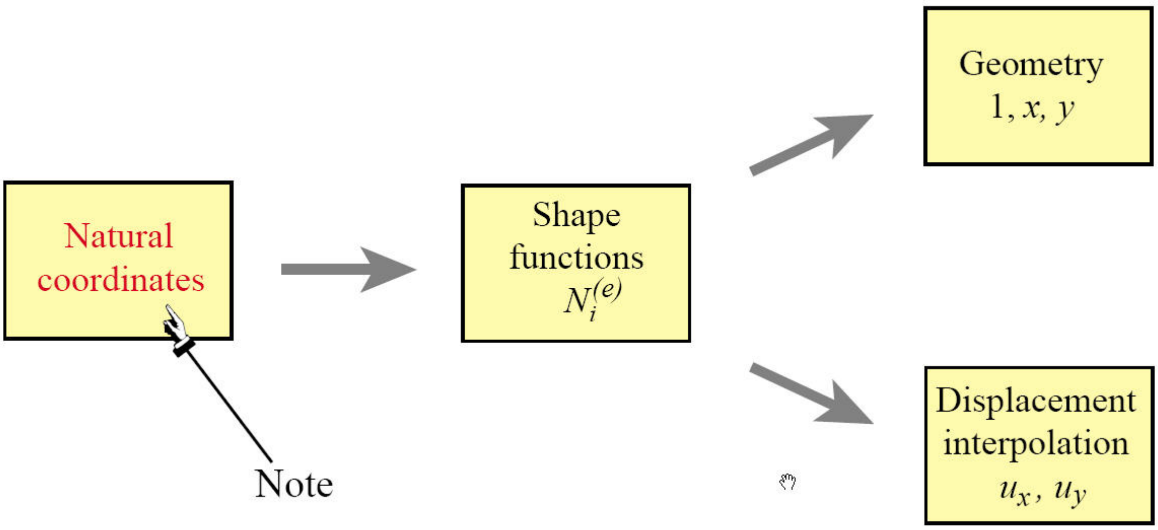
001 REPRESENTACION "ISOPARAMETRICA" DEL ELEMENTO TRIANGULAR LINEAL CURSO 2004-5

Isoparametric Representation (Iso=Equal) for Triangular Elements



001 REPRESENTACION "ISOPARAMETRICA" PARA CUALQUIER ELEMENTO 2D CURSO 2004-5

Isoparametric Representation for *any* 2D Element



Iso-P Representation of 2D Plane Stress Elements with n Nodes

Element Geometry:

$$1 = \sum_{i=1}^{n} N_i^{(e)}, \qquad x = \sum_{i=1}^{n} x_i N_i^{(e)}, \qquad y = \sum_{i=1}^{n} y_i N_i^{(e)}$$

Displacement Interpolation

$$u_x = \sum_{i=1}^{n} u_{xi} N_i^{(e)}, \qquad u_y = \sum_{i=1}^{n} u_{yi} N_i^{(e)}$$

Matrix Form of Above

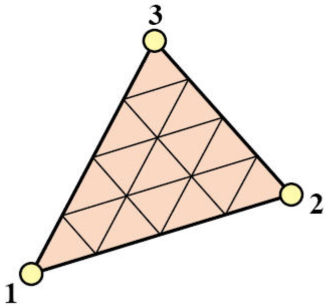
$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ \vdots \\ N_n^{(e)} \end{bmatrix}$$

More Rows May be Added to Interpolate other Quantities from Node Values

thickness h
temperature T

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \\ h \\ T \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \\ h_1 & h_2 & \dots & h_n \\ T_1 & T_2 & \dots & T_n \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ \vdots \\ N_n^{(e)} \end{bmatrix}.$$

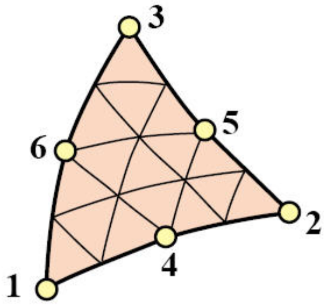
The Linear Triangle



$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \end{bmatrix}$$

$$N_1^{(e)} = \zeta_1, \qquad N_2^{(e)} = \zeta_2, \qquad N_3^{(e)} = \zeta_3$$

The Quadratic Triangle



$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \\ N_4^{(e)} \\ N_5^{(e)} \\ N_6^{(e)} \end{bmatrix}$$

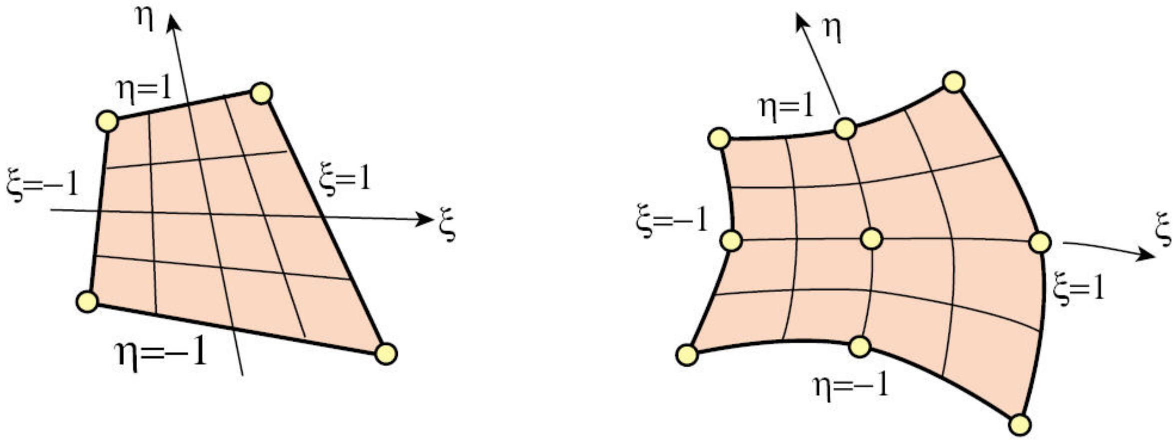
$$N_1^{(e)} = \zeta_1(2\zeta_1 - 1) \qquad N_4^{(e)} = 4\zeta_1\zeta_2$$

$$N_2^{(e)} = \zeta_2(2\zeta_2 - 1) \qquad N_5^{(e)} = 4\zeta_2\zeta_3$$

$$N_3^{(e)} = \zeta_3(2\zeta_3 - 1) \qquad N_6^{(e)} = 4\zeta_3\zeta_1$$

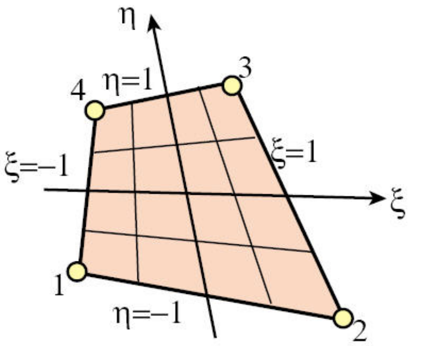
001 COORDENADAS DEL CUADRILATERO CURSO 2004-5

Quadrilateral Coordinates ξ, η



001 REPRESENTACION ISOPARAMETRICA DEL CUADRILATERIO BILINEAL DE 4 NODOS CURSO 2004-5

4-Node Bilinear Quadrilateral

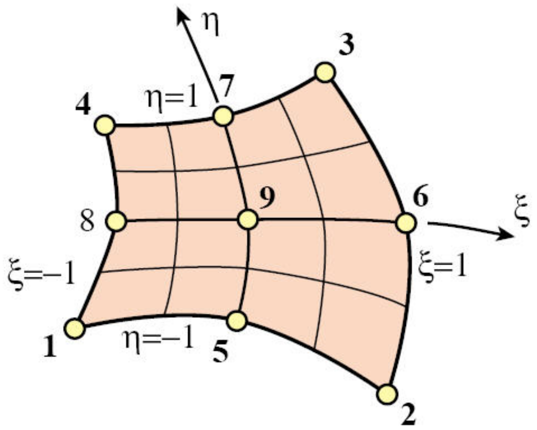


$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \\ N_4^{(e)} \end{bmatrix}$$

$$N_1^{(e)} = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$N_2^{(e)} = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_3^{(e)} = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$N_4^{(e)} = \tfrac{1}{4}(1-\xi)(1+\eta)$$

001 REPRESENTACION ISOPARAMETRICA DEL CUADRILATERIO BICUADRATICO DE 9 NODOS CURSO 2004-5

9 Node Biquadratic Quadrilateral



$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} & u_{x7} & u_{x8} & u_{x9} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} & u_{y7} & u_{y8} & u_{y9} \end{bmatrix} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ \vdots \\ N_9^{(e)} \end{bmatrix}$$

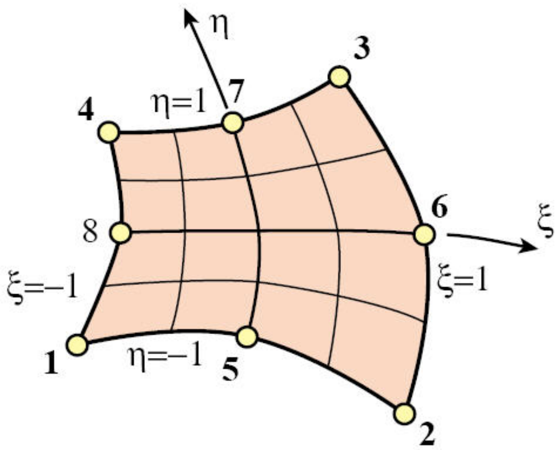
$$N_1^{(e)} = \tfrac{1}{4}(1-\xi)(1-\eta)\xi\eta \qquad N_5^{(e)} = -\tfrac{1}{2}(1-\xi^2)(1-\eta)\eta$$

$$N_2^{(e)} = -\tfrac{1}{4}(1+\xi)(1-\eta)\xi\eta \qquad N_6^{(e)} = \tfrac{1}{2}(1+\xi)(1-\eta^2)\xi \qquad N_9^{(e)} = (1-\xi^2)(1-\eta^2)$$

$$\ldots \qquad \ldots$$

001 CUADRILATERIO “SERENDIPITO” DE 8 NODOS CURSO 2004-5

8 Node "Serendipity" Quadrilateral



Derivation of shape functions is an Exercise in Chapter 18

Exercises

Most Chapters are followed by a list of homework exercises that pose problems of varying difficulty. Each exercise is labeled by a tag of the form

`[type:rating]`

The type is indicated by letters A, C, D or N for exercises to be answered primarily by analytical work, computer programming, descriptive narration, and numerical calculations, respectively. Some exercises involve a combination of these traits, in which case a combination of letters separated by + is used; for example A+N indicates analytical derivation followed by numerical work. For some problems heavy analytical work may be helped by the use of a computer-algebra system, in which case the type is identified as A/C.

The rating is a number between 5 and 50 that estimates the degree of difficulty of an Exercise, in the following "logarithmic" scale:

5 A simple question that can be answered in seconds, or is already answered in the text if the student has read and understood the material.

10 A straightforward question that can be answered in minutes.

15 A relatively simple question that requires some thinking, and may take on the order of half to one hour to answer.

20 Either a problem of moderate difficulty, or a straightforward one requiring lengthy computations or some programming, normally taking one to six hours of work.

25 A scaled up version of the above, estimated to require six hours to one day of work.

30 A problem of moderate difficulty that normally requires on the order of one or two days of work. Arriving at the answer may involve a combination of techniques, some background or reference material, or lenghty but straightforward programming.

40 A difficult problem that may be solvable only by gifted and well prepared individual students, or a team. Difficulties may be due to the need of correct formulation, advanced mathematics, or high level programming. With the proper preparation, background and tools these problems may be solved in days or weeks, while remaining inaccessible to unprepared or average students.

50 A research problem, worthy of publication if solved.

Most Exercises have a rating of 15 or 20. Assigning three or four per week puts a load of roughly 5-10 hours of solution work, plus the time needed to prepare the answer material. Assignments of difficulty 25 or 30 are better handled by groups, or given in take-home exams. Assignments of difficulty beyond 30 are never assigned in the course, but listed as a challenge for an elite group.

Occasionally an Exercise has two or more distinct but related parts identified as items. In that case a rating may be given for each item. For example: [A/C:15+20]. This does not mean that the exercise as a whole has a difficulty of 35, because the scale is roughly logarithmic; the numbers simply rate the expected effort per item.

001	EJERCICIO 1	CURSO 2004-5

[D:10] What is the physical interpretation of the shape-function unit-sum condition discussed in §16.6? Hint: the element must respond exactly in terms of displacements to rigid-body translations in the x and y directions.

001	EJERCICIO 2	CURSO 2004-5

[A:15] Check by algebra that the sum of the shape functions for the six-node quadratic triangle (16.11) is exactly one regardless of natural coordinates values. Hint: show that the sum is expressable as $2S_1^2 - S_1$, where $S_1 = \zeta_1 + \zeta_2 + \zeta_3$.

001	EJERCICIO 3	CURSO 2004-5

[A/C:15] Complete the table of shape functions (16.23) of the nine-node biquadratic quadrilateral. Verify that their sum is exactly one.