

5. Elemento Triangular de TRES NODOS para el Problema de la Tensión Plana.

5.1. Triángulos de Tres Nodos para Tensión Plana - Carlos A. Felippa.

Para observar cómo funciona el Método desde un punto de vista matemático, e introducir los elementos indispensables que permiten que los elementos finitos realicen su trabajo tal y como lo hacen, y de paso profundizar en el uso del programa "Mathematica", en esta sección vamos a presentar el elemento finito más simple de entre los que permiten abordar el problema de la Tensión Plana.

Con esta lección comienza la primera de las dedicadas a la definición de tipos de elementos finitos con los que se puede plantear y resolver el problema mecánico planteado en la lección anterior. En primer lugar se comentan las razones por las que este elemento es tal popular, que fundamentalmente son debidas a la facilidad de generar mallas de triángulos de forma automática. Se comentan seguidamente, aquellos aspectos que distinguirán a este elemento del resto de los que describiremos y utilizaremos. A continuación se procede a su definición, indicando la importancia que el sentido de la numeración de sus nodos tiene en el signo del área calculada con la fórmula que se proporciona. Se presenta el sistema de coordenadas paramétrico que se utilizó en la primera definición histórica que se hizo de él, denominándose a sus coordenadas, coordenadas triangulares (CT). Se comentan los distintos nombres que se han utilizado para referirse a ellas a lo largo del tiempo, y como se definen, especificando la ecuación que relaciona estas coordenadas en cualquier punto del triángulo. Se indica cómo utilizarlas para formular una interpolación lineal de una función dentro del triángulo, a partir de los valores de esa función en los nodos. Se presentan: (1) las ecuaciones que permiten transformar las coordenadas triangulares en cartesianas; (2) como obtener las derivadas parciales en dicha transformación; (3) como obtener las derivadas parciales cartesianas de una función definida en coordenadas triangulares. Con todo lo citado anteriormente, está todo listo para formular matemática el ELEMENTO TRIANGULAR LINEAL. Seguidamente se indica: (1) que la formulación de este elemento las funciones de forma son directamente las coordenadas triangulares; (2) como se realiza la interpolación de los desplazamientos; (3) como obtener las deformaciones a partir de los desplazamientos en los nodos; (4) como obtener las tensiones a partir de los citados desplazamientos; (5) como calcular la matriz de rigidez del elemento, el aspecto de dicha matriz de rigidez cuando el espesor es constante, y que elemento formuló Turner, en su famoso artículo, primero en el que se formuló un elemento finito; y (6) como calcular el vector de fuerzas nodales consistentes, citando la fórmula que permite agilizar el cálculo de integrales en función de las CT.

CHAPTER 15. Triángulos de Tres Nodos para Tensión Plana.
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En este capítulo se proponen como ejercicios los siguientes: (1) obtener la matriz de rigidez de un elemento triangular, en el que su espesor está definido utilizando como funciones de interpolación las coordenadas triangulares, a partir de los valores del espesor conocido en los nodos; (2) un ejercicio que pretende familiarizar al alumno con el uso de la función que permite calcular de forma exacta las integrales planteadas en CT, en triángulos de lados rectos; (3) cálculo del vector de fuerzas nodales consistentes en un triángulo cuyo espesor está definido en función de las CT, equivalente a una fuerza por unidad de volumen; (4) cálculo del vector de fuerzas nodales consistentes en un triángulo de lados rectos, en uno de cuyo lados actúa una fuerza, tanto en x como en y, definida como interpolación mediante las CT; (5) obtención de los elementos de la matriz de rigidez de un elemento triangular dado, definido mediante sus coordenadas cartesianas de sus nodos, su matriz de material, y su espesor, constante. Se propone que una vez resuelto se programe con MATHEMATICA; (6) comprobación de la suma de una serie de filas y columnas de la matriz calculada en el ejercicio anterior, dando una explicación del porqué; (7) Se propone para ser comprobada con Mathematica una solución exacta de la integral sobre un triángulo de lado rectos de una función polinómica definida en CT; y (7) Se propone que se estudie y se utilice al módulo que define un elemento triangular lineal en Mathematica, que se haga un diagrama jerárquico, explicando el propósito de cada celda.

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The Linear Plane Stress Triangle

15-1

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§15.1. INTRODUCTION

This Chapter presents the element equations of a three-node triangle with assumed linear displacements for the plane stress problem formulated in Chapter 14. This element is called the *linear triangle*. This particular element is distinguished in several respects:

- (1) It belongs to both the isoparametric and superparametric element families, which are covered in the next Chapter.
- (2) It allows closed form derivations for its stiffness and consistent forces without the need for numerical integration.
- (3) It cannot be improved by the addition of internal degrees of freedom.

In addition the linear triangle has historical importance.¹ Although it is not a good performer for structural stress analysis, it is still used in problems that do not require high accuracy, as well as in non-structural applications. One reason is that triangular meshes are easily generated over arbitrary domains using techniques such as Delaunay triangulations.

§15.1.1. Parametric Representation of Functions

The concept of *parametric representation* of functions is crucial in modern FEM presentations. Together with numerical integration, it has become a key tool for the systematic development of elements in two and three space dimensions. Without these two tools the element developer would become lost in an algebraic maze as element geometric shapes get more complicated.

The essentials of the idea of parametric representation can be illustrated through a simple example. Consider the following alternative representations of the unit-circle function, $x^2 + y^2 = 1$:

$$y = \sqrt{1 - x^2} \quad (15.1)$$

$$x = \cos \theta, \quad y = \sin \theta \quad (15.2)$$

The direct representation (15.1) fits the conventional function notation, i.e., $y = f(x)$. Given a value of x , it returns one or more y . On the other hand, the representation (15.2) is parametric: both x and y are given in terms of one parameter, the angle θ . Elimination of θ through the trigonometric identity $\cos^2 \theta + \sin^2 \theta = 1$ recovers $x^2 + y^2 = 1$. But there are many situations in which working with the parametric form throughout the development is more convenient. Continuum finite elements provide a striking illustration of this point.

§15.2. TRIANGLE GEOMETRY AND COORDINATE SYSTEMS

The geometry of the 3-node triangle shown in Figure 15.1(a) is specified by the location of its three corner nodes on the $\{x, y\}$ plane. The nodes are labelled 1, 2, 3 while traversing the sides in *counterclockwise* fashion. The location of the corners is defined by their Cartesian coordinates: $\{x_i, y_i\}$ for $i = 1, 2, 3$.

¹ This was one of the two continuum elements presented by Martin, Turner, Clough and Topp in their landmark 1956 paper. Its publication is widely regarded as the start of the present FEM. See Appendix H.

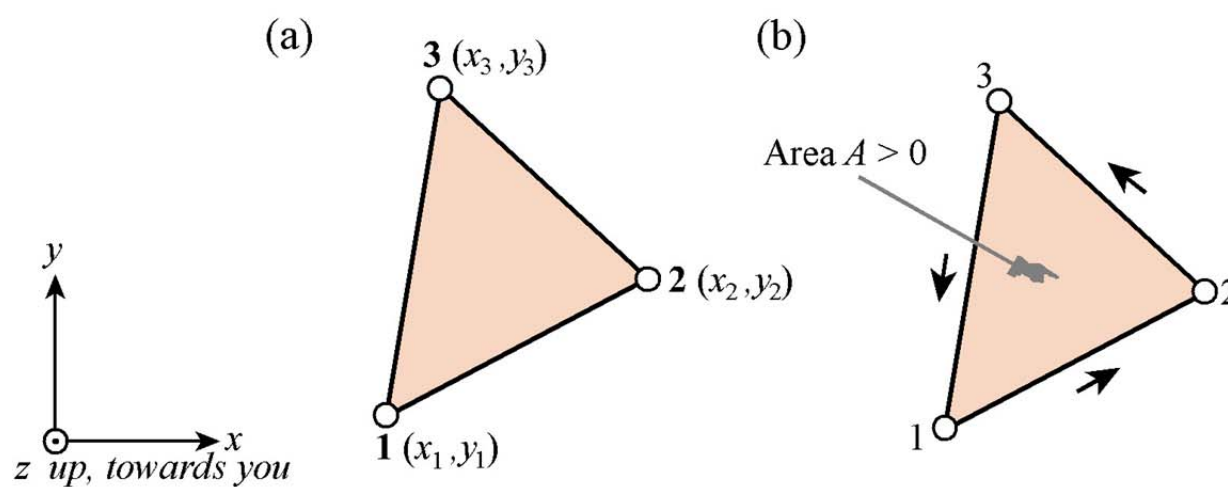


Figure 15.1. The three-node, linear-displacement plane stress triangular element: (a) geometry; (b) area and positive boundary traversal.

The element has six degrees of freedom, defined by the six nodal displacement components $\{u_{xi}, u_{yi}\}$, for $i = 1, 2, 3$. The interpolation of the internal displacements $\{u_x, u_y\}$ from these six values is studied in §15.3, after triangular coordinates are introduced.

The area of the triangle is denoted by A and is given by

$$2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1). \quad (15.3)$$

The area given by (15.3) is a *signed* quantity. It is positive if the corners are numbered in cyclic counterclockwise order as shown in Figure 15.1(b). This convention is followed in the sequel.

§15.2.1. Triangular Coordinates

Points of the triangle may also be located in terms of a *parametric* coordinate system:

$$\zeta_1, \zeta_2, \zeta_3. \quad (15.4)$$

In the literature these three parameters receive an astonishing number of names, as the list given in Table 15.1 shows. In the sequel the name *triangular coordinates* will be used to emphasize the close association with this particular geometry.

Table 15.1 Names of element parametric coordinates in the FEM literature

<i>Name</i>	<i>Applicable to</i>
natural coordinates	all elements
isoparametric coordinates	isoparametric elements
shape function coordinates	isoparametric elements
barycentric coordinates	triangles, tetrahedra
Möbius coordinates	triangles
triangular coordinates	all triangles
area coordinates	straight-sided triangles

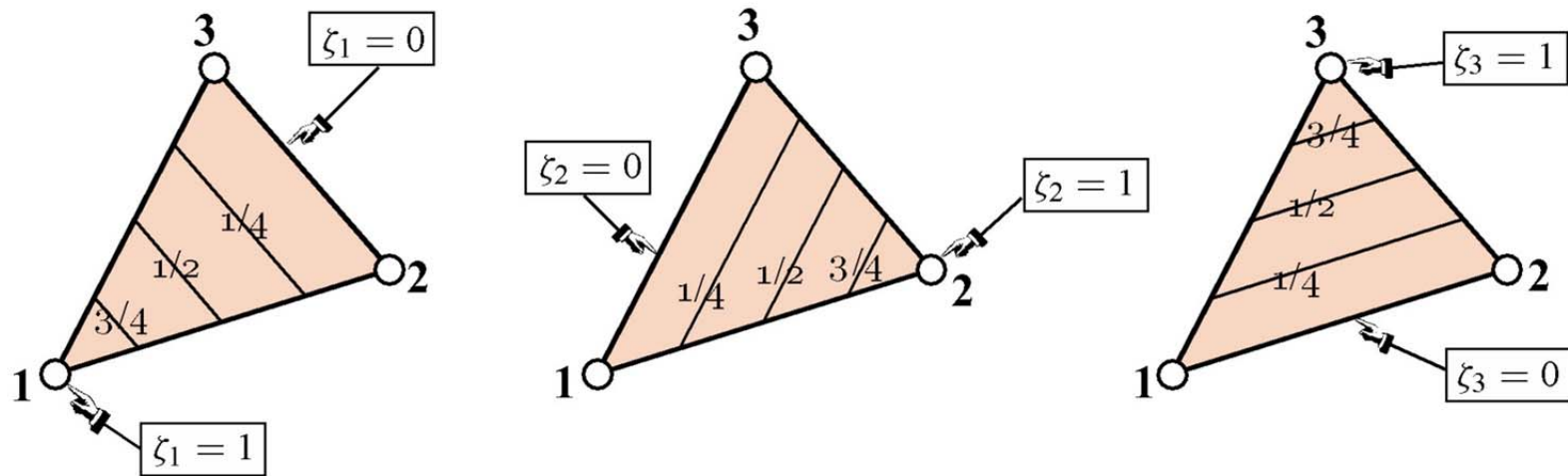


Figure 15.2. Triangular coordinates.

Equations

$$\zeta_i = \text{constant} \tag{15.5}$$

represent a set of straight lines parallel to the side opposite to the i^{th} corner, as depicted in Figure 15.2. The equation of sides 1-2, 2-3 and 3-1 are $\zeta_1 = 0$, $\zeta_2 = 0$ and $\zeta_3 = 0$, respectively. The three corners have coordinates $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. The three midpoints of the sides have coordinates $(\frac{1}{2}, \frac{1}{2}, 0)$, $(0, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2})$, the centroid has coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and so on. The coordinates are not independent because their sum is unity:

$$\zeta_1 + \zeta_2 + \zeta_3 = 1. \tag{15.6}$$

REMARK 15.1

In older (pre-1970) FEM publications triangular coordinates are often called *area coordinates*. This name comes from the following interpretation: $\zeta_i = A_{jk}/A$, where A_{jk} is the area subtended by the triangle formed by the point P and corners j and k , in which j and k are cyclic permutations of i . Historically this was the way the coordinates were defined in 1960s papers. Unfortunately the interpretation does not carry over to general isoparametric triangles with curved sides and thus it is not used here.

§15.2.2. Linear Interpolation

Consider a function $f(x, y)$ that varies *linearly* over the triangle domain. In terms of Cartesian coordinates it may be expressed as

$$f(x, y) = a_0 + a_1x + a_2y, \tag{15.7}$$

where a_0 , a_1 and a_2 are coefficients to be determined from three conditions. In finite element work such conditions are often the *nodal values* taken by f at the corners:

$$f_1, f_2, f_3 \tag{15.8}$$

The expression in triangular coordinates makes direct use of those three values:

$$f(\zeta_1, \zeta_2, \zeta_3) = f_1\zeta_1 + f_2\zeta_2 + f_3\zeta_3 = [f_1 \quad f_2 \quad f_3] \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = [\zeta_1 \quad \zeta_2 \quad \zeta_3] \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}. \quad (15.9)$$

Expression (15.9) is called a *linear interpolant* for f .

§15.2.3. Coordinate Transformations

Quantities that are closely linked with the element geometry are naturally expressed in triangular coordinates. On the other hand, quantities such as displacements, strains and stresses are often expressed in the Cartesian system x, y . We therefore need transformation equations through which we can pass from one coordinate system to the other.

Cartesian and triangular coordinates are linked by the relation

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}. \quad (15.10)$$

The first equation says that the sum of the three coordinates is one. The next two express x and y linearly as homogeneous forms in the triangular coordinates. These apply the interpolant formula (15.9) to the Cartesian coordinates: $x = x_1\zeta_1 + x_2\zeta_2 + x_3\zeta_3$ and $y = y_1\zeta_1 + y_2\zeta_2 + y_3\zeta_3$.

Inversion of (15.10) yields

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} 2A_{23} & y_{23} & x_{32} \\ 2A_{31} & y_{31} & x_{13} \\ 2A_{12} & y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}. \quad (15.11)$$

Here $x_{jk} = x_j - x_k$, $y_{jk} = y_j - y_k$, A is the triangle area given by (15.3) and A_{jk} denotes the area subtended by corners j, k and the origin of the x - y system. If this origin is taken at the centroid of the triangle, $A_{23} = A_{31} = A_{12} = A/3$.

§15.2.4. Partial Derivatives

From equations (15.10) and (15.11) we immediately obtain the following relations between partial derivatives:

$$\frac{\partial x}{\partial \zeta_i} = x_i, \quad \frac{\partial y}{\partial \zeta_i} = y_i, \quad (15.12)$$

$$2A \frac{\partial \zeta_i}{\partial x} = y_{jk}, \quad 2A \frac{\partial \zeta_i}{\partial y} = x_{kj}. \quad (15.13)$$

In (15.13) j and k denote the *cyclic permutations* of i . For example, if $i = 2$, then $j = 3$ and $k = 1$.

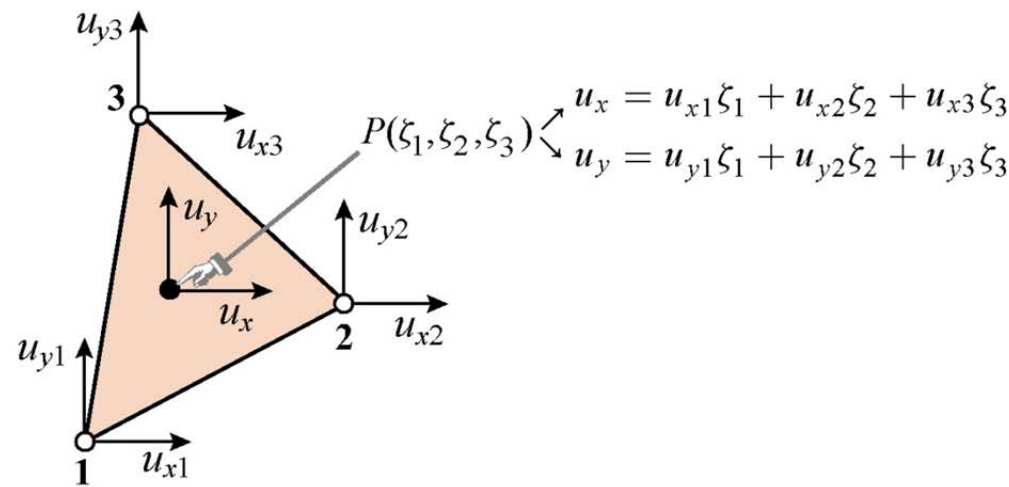


Figure 15.3. Displacement interpolation over triangle.

The derivatives of a function $f(\zeta_1, \zeta_2, \zeta_3)$ with respect to x or y follow immediately from (15.13) and application of the chain rule:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{2A} \left(\frac{\partial f}{\partial \zeta_1} y_{23} + \frac{\partial f}{\partial \zeta_2} y_{31} + \frac{\partial f}{\partial \zeta_3} y_{12} \right) \\ \frac{\partial f}{\partial y} &= \frac{1}{2A} \left(\frac{\partial f}{\partial \zeta_1} x_{32} + \frac{\partial f}{\partial \zeta_2} x_{13} + \frac{\partial f}{\partial \zeta_3} x_{21} \right) \end{aligned} \tag{15.14}$$

which in matrix form is

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \zeta_1} \\ \frac{\partial f}{\partial \zeta_2} \\ \frac{\partial f}{\partial \zeta_3} \end{bmatrix} \tag{15.15}$$

With these mathematical ingredients in place we are now in a position to handle the derivation of straight-sided triangular elements, and in particular the linear triangle.

§15.3. ELEMENT DERIVATION

The simplest triangular element for plane stress (and in general, for 2D problems of variational index $m = 1$) is the three-node triangle with *linear shape functions*. The shape functions are simply the triangular coordinates. That is, $N_i^{(e)} = \zeta_i$ for $i = 1, 2, 3$.

§15.3.1. Displacement Interpolation

For the plane stress problem we select the linear interpolation (15.9) for the displacement components u_x and u_y at an arbitrary point $P(\zeta_1, \zeta_2, \zeta_3)$, as illustrated in Figure 15.3:

$$\begin{aligned} u_x &= u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3, \\ u_y &= u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3. \end{aligned} \tag{15.16}$$

These relations can be combined in a matrix form that befits the general expression (14.17) for an arbitrary plane stress element:

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{N}^{(e)} \mathbf{u}^{(e)}. \quad (15.17)$$

§15.3.2. Strain-Displacement Equations

The strains within the elements are obtained by differentiating the shape functions with respect to x and y . Using (15.14) and the general form (14.18) we get

$$\mathbf{e} = \mathbf{D}\mathbf{N}^{(e)}\mathbf{u}^{(e)} = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{B}\mathbf{u}^{(e)}, \quad (15.18)$$

in which \mathbf{D} denotes the symbolic strain-to-displacement differentiation operator given in (14.6).

Note that the strains are *constant* over the element. This is the origin of the name *constant strain triangle* (CST) given it in many finite element publications.

§15.3.3. Stress-Strain Equations

The stress field $\boldsymbol{\sigma}$ is related to the strain field by the elastic constitutive equation in (14.5), which is repeated here for convenience:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{E}\mathbf{e}, \quad (15.19)$$

where E_{ij} are plane stress elastic moduli. The constitutive matrix \mathbf{E} will be assumed to be constant over the element. Because the strains are constant, so are the stresses.

§15.3.4. The Stiffness Matrix

The element stiffness matrix is given by the general formula (14.23), which is repeated here for convenience:

$$\mathbf{K}^{(e)} = \int_{\Omega^{(e)}} h \mathbf{B}^T \mathbf{E} \mathbf{B} d\Omega^{(e)}, \quad (15.20)$$

where $\Omega^{(e)}$ is the triangle domain, and h is the plate thickness that appears in the plane stress problem. Since \mathbf{B} and \mathbf{E} are constant, they can be taken out of the integral:

$$\begin{aligned} \mathbf{K}^{(e)} &= \mathbf{B}^T \mathbf{E} \mathbf{B} \int_{\Omega^{(e)}} h d\Omega^{(e)} \\ &= \frac{1}{4A^2} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \int_{\Omega^{(e)}} h d\Omega^{(e)}. \end{aligned} \tag{15.21}$$

If the thickness h is uniform over the element the integral in (15.21) is simply hA , and we obtain the closed form

$$\mathbf{K}^{(e)} = Ah \mathbf{B}^T \mathbf{E} \mathbf{B} = \frac{h}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}. \tag{15.22}$$

Exercise 15.1 deals with the case of a linearly varying thickness.

§15.3.5. The Consistent Nodal Force Vector

For simplicity we consider here only internal body forces² defined by the vector field

$$\mathbf{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix} \tag{15.23}$$

which is specified per unit of volume. The consistent nodal force vector $\mathbf{f}^{(e)}$ is given by the general formula (14.23) of the previous Chapter:

$$\mathbf{f}^{(e)} = \int_{\Omega^{(e)}} h (\mathbf{N}^{(e)})^T \mathbf{b} d\Omega^{(e)} = \int_{\Omega^{(e)}} h \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_1 \\ \zeta_2 & 0 \\ 0 & \zeta_2 \\ \zeta_3 & 0 \\ 0 & \zeta_3 \end{bmatrix} \mathbf{b} d\Omega^{(e)}. \tag{15.24}$$

The simplest case is when the body force components (15.23) as well as the thickness h are constant over the element. Then we need the integrals

$$\int_{\Omega^{(e)}} \zeta_1 d\Omega^{(e)} = \int_{\Omega^{(e)}} \zeta_2 d\Omega^{(e)} = \int_{\Omega^{(e)}} \zeta_3 d\Omega^{(e)} = \frac{1}{3}A \tag{15.25}$$

² For consistent force computations corresponding to distributed boundary loads over a side, see Exercise 15.4.

which replaced into (15.24) gives

$$\mathbf{f}^{(e)} = \frac{Ah}{3} \begin{bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{bmatrix}. \quad (15.26)$$

This agrees with the simple force-lumping procedure, which assigns one third of the total force along the $\{x, y\}$ directions: Ahb_x and Ahb_y , to each corner.

REMARK 15.2

The integral (15.25) is a particular case of the general integration formula of monomials in triangular coordinates:

$$\frac{1}{2A} \int_{\Omega^{(e)}} \zeta_1^i \zeta_2^j \zeta_3^k d\Omega^{(e)} = \frac{i! j! k!}{(i + j + k + 2)!}, \quad i \geq 0, j \geq 0, k \geq 0. \quad (15.27)$$

which can be proven by recursive integration by parts. This formula *only holds for triangles with straight sides*, and consequently is useless for higher order curved elements. The result (15.25) is obtained by setting $i = 1, j = k = 0$ in (15.27).

§15.4. *CONSISTENCY VERIFICATION

It remains to check whether the piecewise linear expansion (15.16) for the element displacements meets the completeness and continuity criteria studied in more detail in Chapter 19 for finite element trial functions. Such *consistency* conditions are sufficient to insure convergence towards the exact solution of the mathematical model as the mesh is refined.

The variational index for the plane stress problem is $m = 1$. Consequently the trial functions should be 1-complete, C^0 continuous, and C^1 piecewise differentiable.

§15.4.1. *Checking Completeness

The completeness condition for variational order $m = 1$ require that the shape functions $N_i = \zeta_i$ be able to represent exactly any linear displacement field:

$$u_x = \alpha_0 + \alpha_1x + \alpha_2y, \quad u_y = \beta_0 + \beta_1x + \beta_2y. \quad (15.28)$$

To check this we obtain the nodal values associated with the motion (15.28)

$$\left. \begin{aligned} u_{xi} &= \alpha_0 + \alpha_1x_i + \alpha_2y_i \\ u_{yi} &= \beta_0 + \beta_1x_i + \beta_2y_i \end{aligned} \right\} \quad i = 1, 2, 3, \quad (15.29)$$

replace them into (15.16) and see if we recover (15.28). Here are the detailed calculations for component u_x :

$$\begin{aligned} u_x &= \sum_i u_{xi} \zeta_i = \sum_i (\alpha_0 + \alpha_1x_i + \alpha_2y_i) \zeta_i = \sum_i (\alpha_0 \zeta_i + \alpha_1x_i \zeta_i + \alpha_2y_i \zeta_i) \\ &= \alpha_0 \sum_i \zeta_i + \alpha_1 \sum_i (x_i \zeta_i) + \alpha_2 \sum_i (y_i \zeta_i) = \alpha_0 + \alpha_1x + \alpha_2y. \end{aligned} \quad (15.30)$$

Component u_y can be similarly checked. Consequently (15.16) satisfies the completeness requirement for the plane stress problem (and in general, for any problem of variational index 1).

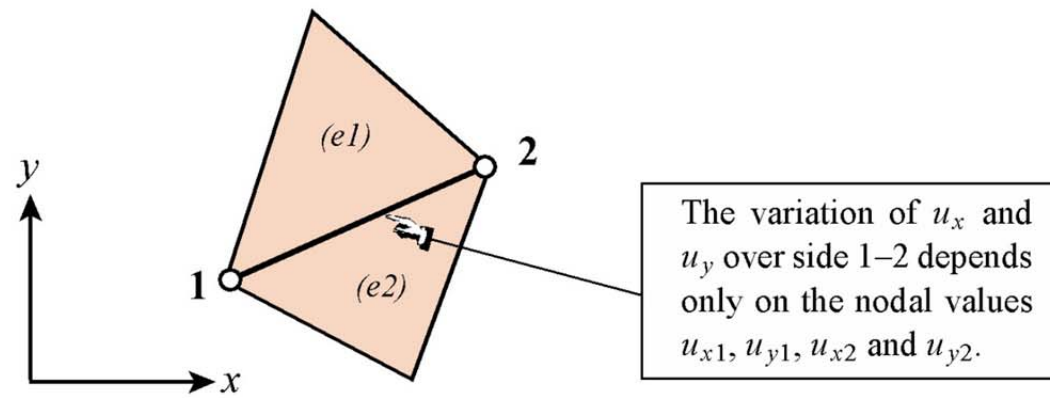


Figure 15.4. Interelement continuity check.

§15.4.2. *Checking Continuity

Along any triangle side, the variation of u_x and u_y is *linear and uniquely determined by the value at the nodes on that side*. For example, over the side 1-2 of an individual triangle:

$$\begin{aligned} u_x &= u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3 = u_{x1}\zeta_1 + u_{x2}\zeta_2, \\ u_y &= u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3 = u_{y1}\zeta_1 + u_{y2}\zeta_2. \end{aligned} \tag{15.31}$$

because $\zeta_3 = 0$ along that side.

An identical argument holds for that side when it belongs to an adjacent triangle, such as elements (e1) and (e2) shown in Figure 15.4. Therefore, if the values on all elements that meet at a node are the same, u_x and u_y match along the side, and the trial function is C^0 continuous across elements. Because the functions are continuous inside the elements, it follows that the conformity requirement is met.

Finally, a piecewise linear trial function is obviously C^1 piecewise differentiable and consequently has finite energy. Thus the two continuity requirements are satisfied.

§15.5. *THE TONTI DIAGRAM OF THE LINEAR TRIANGLE

For further developments covered in advanced FEM course, it is convenient to split the governing equations of the element. In the case of the linear triangle they are, omitting element superscripts:

$$\epsilon = \mathbf{B}\mathbf{u}, \quad \sigma = \mathbf{E}\epsilon, \quad \mathbf{f} = \mathbf{A}^T \sigma = V\mathbf{B}^T \sigma. \tag{15.32}$$

in which $V = h_m A$ is the volume of the element, h_m being the mean thickness.

The equations (15.32) may be represented with the matrix diagram shown in Figure 15.5.

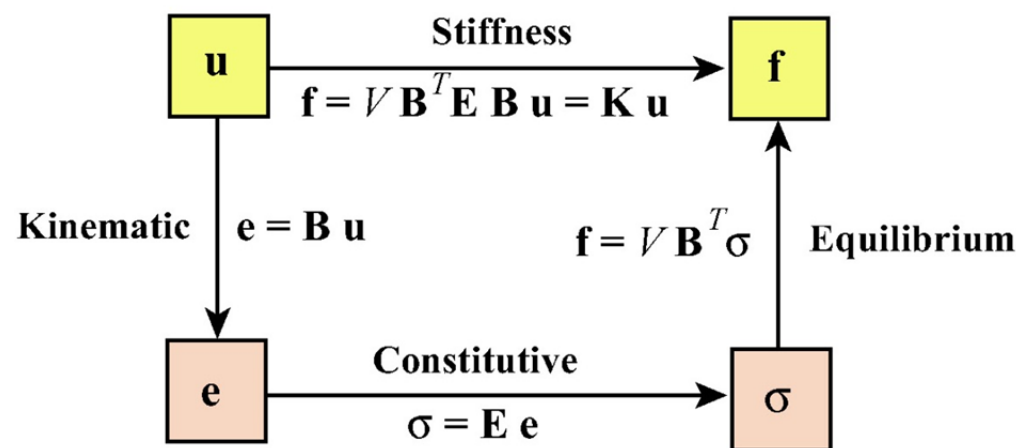


Figure 15.5. Tonti matrix diagram for the linear triangle.

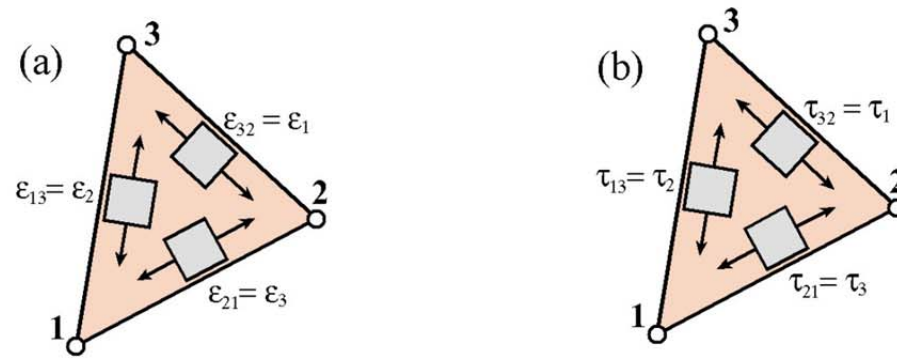


Figure 15.6. Intrinsic strain and stress fields for the 3-node linear triangle: (a) natural strains ϵ_i , (b) natural stresses τ_i .

§15.6. *DERIVATION USING NATURAL STRAINS AND STRESSES

The element derivation in §15.3 uses Cartesian strains and stresses, as well as $x - y$ displacements. The only intrinsic quantities are the triangle coordinates. This subsection examines the derivation of the element stiffness matrix through natural strains, natural stresses and covariant displacements. Although the procedure does not offer obvious shortcuts over the previous derivation, it becomes important in the construction of more complicated high performance elements. It also helps reading recent literature.

§15.6.1. *Natural Strains and Stresses

Natural strains are extensional strains directed parallel to the triangle sides, as shown in Figure 15.6(a). They are denoted by $\epsilon_{21} \equiv \epsilon_3$, $\epsilon_{32} \equiv \epsilon_1$, and $\epsilon_{13} \equiv \epsilon_2$. Because they are constant over the triangle, no node value association is needed.

Natural stresses are normal stresses directed parallel to the triangle sides, as shown in Figure 15.6(b). They are denoted by $\tau_{21} \equiv \tau_3$, $\tau_{32} \equiv \tau_1$, and $\tau_{13} \equiv \tau_2$. Because they are constant over the triangle, no node value association is needed.

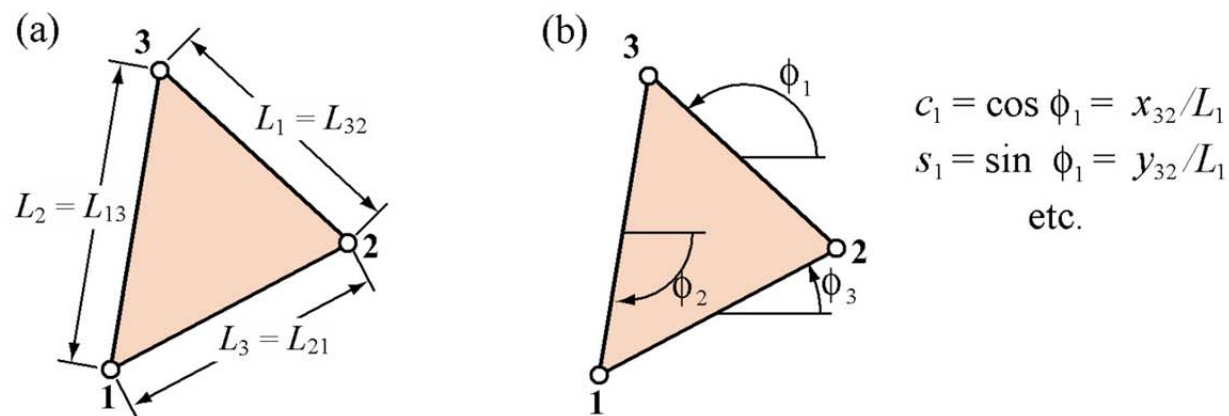
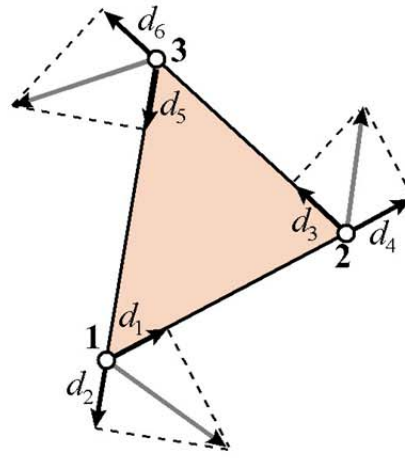


Figure 15.7. Quantities appearing in natural strain and stress calculations: (a) side lengths, (b) side directions.

The natural strains can be related to Cartesian strains by the tensor transformation³

$$\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \begin{bmatrix} c_1^2 & s_1^2 & s_1 c_1 \\ c_2^2 & s_2^2 & s_2 c_2 \\ c_3^2 & s_3^2 & s_3 c_3 \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{T}_e^{-1} \mathbf{e}. \tag{15.33}$$

³ This is the “straingage rosette” transformation studied in Mechanics of Materials books.


Figure 15.8. Covariant node displacements d_i .

Here $c_1 = x_{32}/L_1$, $s_1 = y_{32}/L_1$, $c_2 = x_{13}/L_2$, $s_2 = y_{13}/L_2$, $c_3 = x_{21}/L_3$, and $s_3 = y_{21}/L_3$, are sine/cosines of the side directions with respect to $\{x, y\}$, as illustrated in Figure 15.7. The inverse of this relation is

$$\mathbf{e} = \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \frac{1}{4A^2} \begin{bmatrix} y_{31}y_{21}L_1^2 & y_{12}y_{32}L_2^2 & y_{23}y_{13}L_3^2 \\ x_{31}x_{21}L_1^2 & x_{12}x_{32}L_2^2 & x_{23}x_{13}L_3^2 \\ (y_{31}x_{12} + x_{13}y_{21})L_1^2 & (y_{12}x_{23} + x_{21}y_{32})L_2^2 & (y_{23}x_{31} + x_{32}y_{13})L_3^2 \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \mathbf{T}_e \boldsymbol{\epsilon}. \quad (15.34)$$

Note that \mathbf{T}_e is constant over the triangle.

From the invariance of the strain energy density $\boldsymbol{\sigma}^T \mathbf{e} = \boldsymbol{\tau}^T \boldsymbol{\epsilon}$ it follows that the stresses transform as $\boldsymbol{\tau} = \mathbf{T}_e \boldsymbol{\sigma}$ and $\boldsymbol{\sigma} = \mathbf{T}_e^{-1} \boldsymbol{\tau}$. That strain energy density may be expressed as

$$\mathcal{U} = \frac{1}{2} \mathbf{e}^T \mathbf{E} \mathbf{e} = \frac{1}{2} \boldsymbol{\epsilon}^T \mathbf{E}_n \boldsymbol{\epsilon}, \quad \mathbf{E}_n = \mathbf{T}_e^T \mathbf{E} \mathbf{T}_e. \quad (15.35)$$

Here \mathbf{E}_n is a stress-strain matrix that relates natural stresses to natural strains as $\boldsymbol{\tau} = \mathbf{E}_n \boldsymbol{\epsilon}$. It may be therefore called the natural constitutive matrix.

§15.6.2. *Covariant Node Displacements

Covariant node displacements d_i are directed along the side directions, as shown in Figure 15.8, which defines the notation used for them. The natural strains are evidently given by the relations $\epsilon_1 = (d_6 - d_3)/L_1$, $\epsilon_2 = (d_2 - d_5)/L_2$ and $\epsilon_3 = (d_4 - d_1)/L_3$, which collected in matrix form are

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1/L_1 & 0 & 0 & 1/L_1 \\ 0 & 1/L_2 & 0 & 0 & -1/L_2 & 0 \\ -1/L_3 & 0 & 0 & 1/L_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} = \mathbf{B}_\epsilon \mathbf{d}. \quad (15.36)$$

The covariant node displacements are related to the Cartesian node displacements by

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} = \begin{bmatrix} c_3 & s_3 & 0 & 0 & 0 & 0 \\ c_2 & s_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & s_1 & 0 & 0 \\ 0 & 0 & c_3 & s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_2 & s_2 \\ 0 & 0 & 0 & 0 & c_1 & s_1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{T}_d \mathbf{u}. \quad (15.37)$$

The inverse relation is

$$\mathbf{u} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} L_3 y_{31} & L_2 y_{21} & 0 & 0 & 0 & 0 \\ L_3 x_{13} & L_2 x_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & L_1 y_{12} & L_3 y_{32} & 0 & 0 \\ 0 & 0 & L_1 x_{21} & L_3 x_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & L_2 y_{23} & L_1 y_{13} \\ 0 & 0 & 0 & 0 & L_2 x_{32} & L_1 x_{31} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} = \mathbf{T}_d^{-1} \mathbf{d}. \quad (15.38)$$

§15.6.3. *The Natural Stiffness Matrix

The natural stiffness matrix for constant h is

$$\mathbf{K}_n = (Ah) \mathbf{B}_e^T \mathbf{E}_n \mathbf{B}_e, \quad \mathbf{E}_n = \mathbf{T}_e^T \mathbf{E} \mathbf{T}_e. \quad (15.39)$$

The Cartesian stiffness matrix is

$$\mathbf{K} = \mathbf{T}_d^T \mathbf{K}_n \mathbf{T}_d. \quad (15.40)$$

Comparing with $\mathbf{K} = (Ah) \mathbf{B}^T \mathbf{E} \mathbf{B}$ we see that

$$\mathbf{B} = \mathbf{T}_e \mathbf{B}_e \mathbf{T}_d, \quad \mathbf{B}_e = \mathbf{T}_e^{-1} \mathbf{B} \mathbf{T}_d^{-1}. \quad (15.41)$$

Homework Exercises for Chapter 15
The Linear Plane Stress Triangle

EXERCISE 15.1

[A:15] Assume that the 3-node plane stress triangle has *variable* thickness defined over the element by the linear interpolation formula

$$h(\zeta_1, \zeta_2, \zeta_3) = h_1\zeta_1 + h_2\zeta_2 + h_3\zeta_3, \tag{E15.1}$$

where h_1, h_2 and h_3 are the thicknesses at the corner nodes. Show that the element stiffness matrix is still given by (15.22) but with h replaced by the mean thickness $h_m = (h_1 + h_2 + h_3)/3$. *Hint:* use (15.21) and (15.27).

EXERCISE 15.2

[A:20] The exact integrals of triangle-coordinate monomials over a straight-sided triangle are given by the formula

$$\frac{1}{2A} \int_A \zeta_1^i \zeta_2^j \zeta_3^k dA = \frac{i! j! k!}{(i + j + k + 2)!} \tag{E15.2}$$

where A denotes the area of the triangle, and i, j and k are nonnegative integers. Tabulate the right-hand side for combinations of exponents i, j and k such that $i + j + k \leq 3$, beginning with $i = j = k = 0$. Remember that $0! = 1$. (*Labor-saving hint:* don't bother repeating exponent permutations; for example $i = 2, j = 1, k = 0$ and $i = 1, j = 2, k = 0$ are permutations of the same thing. Hence one needs to tabulate only cases in which $i \geq j \geq k$).

EXERCISE 15.3

[A/C:20] Compute the consistent node force vector $\mathbf{f}^{(e)}$ for body loads over a linear triangle, if the element thickness varies as per (E15.1), $b_x = 0$, and $b_y = b_{y1}\zeta_1 + b_{y2}\zeta_2 + b_{y3}\zeta_3$. Check that for $h_1 = h_2 = h_3 = h$ and $b_{y1} = b_{y2} = b_{y3} = b_y$ you recover (15.26). For the integrals over the triangle area use the formula (E15.2).

Partial result: $f_{y1} = (A/60)[b_{y1}(6h_1 + 2h_2 + 2h_3) + b_{y2}(2h_1 + 2h_2 + h_3) + b_{y3}(2h_1 + h_2 + 2h_3)]$.

EXERCISE 15.4

[A/C:20] Derive the formula for the consistent force vector $\mathbf{f}^{(e)}$ of a linear triangle of constant thickness h , if side 1-2 ($\zeta_3 = 0, \zeta_2 = 1 - \zeta_1$), is subject to a linearly varying boundary force $\mathbf{q} = h\hat{\mathbf{t}}$ such that

$$q_x = q_{x1}\zeta_1 + q_{x2}\zeta_2 = q_{x1}(1 - \zeta_2) + q_{x2}\zeta_2, \quad q_y = q_{y1}\zeta_1 + q_{y2}\zeta_2 = q_{y1}(1 - \zeta_2) + q_{y2}\zeta_2. \tag{E15.3}$$

This “line force” \mathbf{q} has dimension of force per unit of side length.

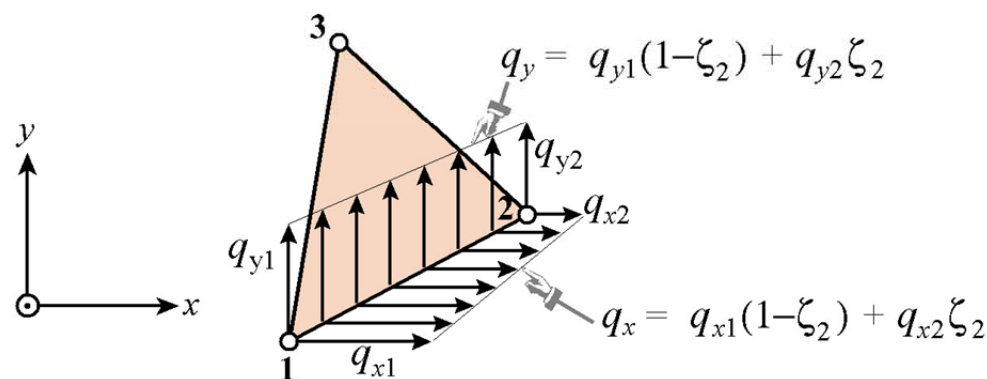


Figure E15.1. Line forces on triangle side 1-2 for Exercise 15.4.

Procedure. Use the last term of the line integral (14.21), in which $\hat{\mathbf{t}}$ is replaced by \mathbf{q}/h , and show that since the contribution of sides 2-3 and 3-1 to the line integral vanish,

$$W^{(e)} = (\mathbf{u}^{(e)})^T \mathbf{f}^{(e)} = \int_{\Gamma^{(e)}} \mathbf{u}^T \mathbf{q} d\Gamma^{(e)} = \int_0^1 \mathbf{u}^T \mathbf{q} L_{21} d\zeta_2, \quad (\text{E15.4})$$

where L_{21} is the length of side 1-2. Replace $u_x(\zeta_2) = u_{x1}(1 - \zeta_2) + u_{x2}\zeta_2$; likewise for u_y, q_x and q_y , integrate and identify with the inner product shown as the second term in (E15.4). Partial result: $f_{x1} = L_{21}(2q_{x1} + q_{x2})/6$, $f_{x3} = f_{y3} = 0$. *Note.* The following *Mathematica* script solves this Exercise. If you decide to use it, explain the logic.

```
ClearAll[ux1,uy1,ux2,uy2,ux3,uy3,z2,L12];
ux=ux1*(1-z2)+ux2*z2; uy=uy1*(1-z2)+uy2*z2;
qx=qx1*(1-z2)+qx2*z2; qy=qy1*(1-z2)+qy2*z2;
We=Simplify[L12*Integrate[qx*ux+qy*uy,{z2,0,1}]];
fe=Table[Coefficient[We,{ux1,uy1,ux2,uy2,ux3,uy3}][[i]],{i,1,6}];
fe=Simplify[fe]; Print["fe=",fe];
```

EXERCISE 15.5

[C+N:15] Compute the entries of $\mathbf{K}^{(e)}$ for the following plane stress triangle:

$$x_1 = 0, y_1 = 0, x_2 = 3, y_2 = 1, x_3 = 2, y_3 = 2, \quad (\text{E15.5})$$

$$\mathbf{E} = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}, \quad h = 1.$$

This may be done by hand (it is a good exercise in matrix multiplication) or (more quickly) using the following *Mathematica* script:

```
Stiffness3NodePlaneStressTriangle[{{x1_,y1_},{x2_,y2_},{x3_,y3_}},
  Emat_,{h_}]:=Module[{x21,x13,x32,y12,y31,y23,A,Be,Ke},
  A=Simplify[(x2*y3-x3*y2+(x3*y1-x1*y3)+(x1*y2-x2*y1))/2];
  {x21,x13,x32}={x2-x1,x1-x3,x3-x2};
  {y12,y31,y23}={y1-y2,y3-y1,y2-y3};
  Be={{y23,0,y31,0,y12,0},{0,x32,0,x13,0,x21},
  {x32,y23,x13,y31,x21,y12}}/(2*A);
  Ke=A*h*Transpose[Be].Emat.Be;Return[Ke];

Ke=Stiffness3NodePlaneStressTriangle[{{0,0},{3,1},{2,2}},
  {{100,25,0},{25,100,0},{0,0,50}},{1}];
Print["Ke=",Ke//MatrixForm];
Print["eigs of Ke=",Chop[Eigenvalues[N[Ke]]]];
Show[Graphics[Line[{{0,0},{3,1},{2,2},{0,0}}]],Axes->True];
```

Check it out: $K_{11} = 18.75$, $K_{66} = 118.75$. The last statement draws the triangle.

EXERCISE 15.6

[A+C:15] Show that the sum of the rows (and columns) 1, 3 and 5 of $\mathbf{K}^{(e)}$ as well as the sum of rows (and columns) 2, 4 and 6 must vanish. Check it with the foregoing script.

EXERCISE 15.7

[A/C:30] Let $p(\zeta_1, \zeta_2, \zeta_3)$ represent a *polynomial* expression in the natural coordinates. The integral

$$\int_{\Omega^{(e)}} p(\zeta_1, \zeta_2, \zeta_3) d\Omega \tag{E15.6}$$

over a straight-sided triangle can be computed symbolically by the following *Mathematica* module:

```
IntegrateOverTriangle[expr_, tcoord_, A_, max_] := Module [{p, i, j, k, z1, z2, z3, c, s=0},
  p=Expand[expr]; {z1, z2, z3}=tcoord;
  For [i=0, i<=max, i++, For [j=0, j<=max, j++, For [k=0, k<=max, k++,
    c=Coefficient[Coefficient[Coefficient[p, z1, i], z2, j], z3, k];
    s+=2*c*(i!*j!*k!)/((i+j+k+2)!);
  ]]];
  Return[Simplify[A*s]] ];
```

This is referenced as `int=IntegrateOverTriangle[p, {z1, z2, z3}, A, max]`. Here p is the polynomial to be integrated, z_1, z_2 and z_3 denote the symbols used for the triangular coordinates, A is the triangle area and max the highest exponent appearing in a triangular coordinate. The module name returns the integral. For example, if $p=16+5*b*z_2^2+z_1^3+z_2*z_3*(z_2+z_3)$ the call `int=IntegrateOverTriangle[p, {z1, z2, z3}, A, 3]` returns `int=A*(97+5*b)/6`. Explain how the module works.

EXERCISE 15.8

[C+D:25] Access the file `Trig3PlaneStress.nb` from the course Web site by clicking on the appropriate link in Chapter 15 Index. This is a *Mathematica* 4.1 Notebook that does plane stress FEM analysis using the 3-node linear triangle.

Download the Notebook into your directory. Load into *Mathematica*. Execute the top 7 input cells (which are actually initialization cells) so the necessary modules are compiled. Each cell is preceded by a short comment cell which outlines the purpose of the modules it holds. Notes: (1) the plot-module cell may take a while to run through its tests; be patient; (2) to get rid of unsightly messages and silly beeps about similar names, initialize each cell twice.

After you are satisfied everything works fine, run the cantilever beam problem, which is defined in the last input cell.

After you get a feel of how this code operate, study the source. Prepare a hierarchical diagram of the modules,⁴ beginning with the main program of the last cell. Note which calls what, and briefly explain the purpose of each module. Return this diagram as answer to the homework. You do not need to talk about the actual run and results; those will be discussed in Part III.

⁴ A hierarchical diagram is a list of modules and their purposes, with indentation to show dependence, similar to the table of contents of a book. For example, if module AAAA calls BBBB and CCCC, and BBBB calld DDDD, the hierarchical diagram may look like:

```
AAAA - purpose of AAAA
  BBBB - purpose of BBBB
    DDDD - purpose of DDDD
  CCCC - purpose of CCCC
```

Section 15: THE LINEAR PLANE STRESS TRIANGLE

15-18

Hint: a hierarchical diagram for Trig3PlaneStress.nb begins like

```
Main program in Cell 8 - drives the FEM analysis
  GenerateNodes - generates node coordinates of regular mesh
  GenerateTriangles - generate element node lists of regular mesh
  .....
```

15-18

