UNIVERSIDAD POLITECNICA DE VALENCIA

DEPARTAMENTO DE INGENIERIA MECANICA Y DE MATERIALES

ELEMENTOS FINITOS

(E.T.S.I.I.V)

FORMULACION DE ELEMENTOS FINITOS

LECCION 4.- ELEMENTO TRAINGULAR LINEAL

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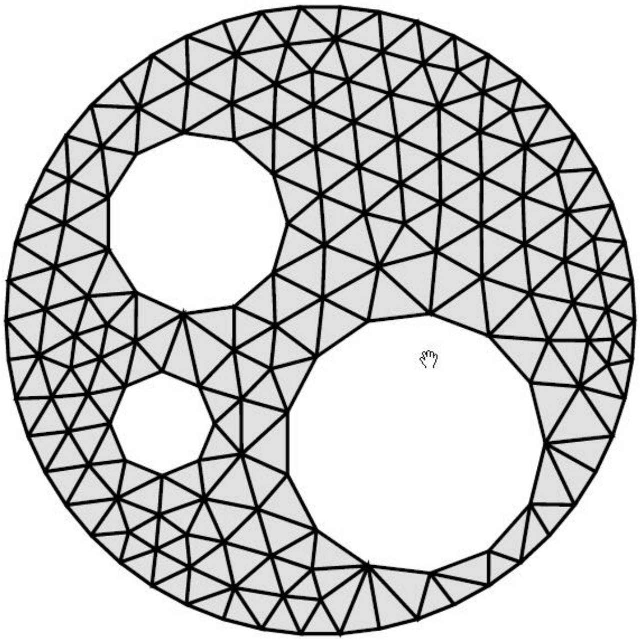
Dr. Ingeniero Industrial

Valencia, 2005

Triangles are Still Popular Because of Geometric Versatility and Ease of Automated Mesh Generation



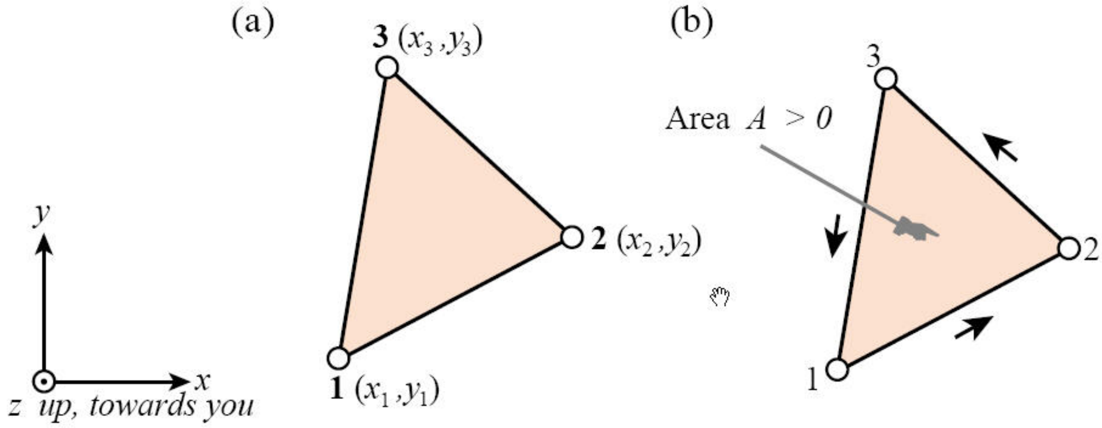
(1) It belongs to both the isoparametric and superparametric element families, which are covered in the next Chapter.

(2) It allows closed form derivations for its stiffness and consistent forces without the need for numerical integration.

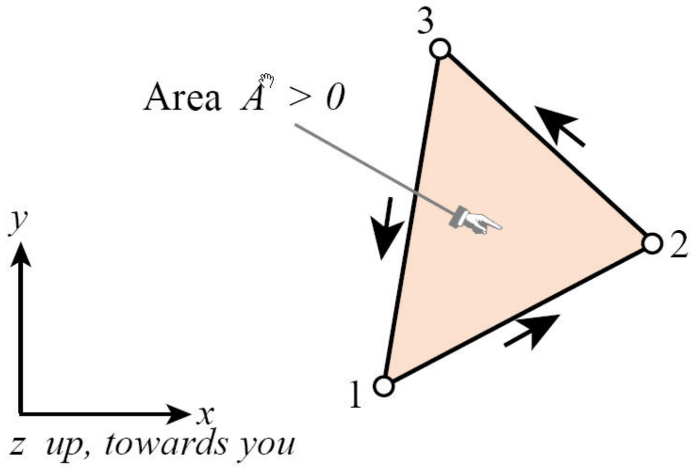
(3) It cannot be improved by the addition of internal degrees of freedom.

In addition the linear triangle has historical importance.[1] Although it is not a good performer for structural stress analysis, it is still used in problems that do not require high accuracy, as well as in non-structural applications. One reason is that triangular meshes are easily generated over arbitrary domains using techniques such as Delaunay triangulations.

The Triangle Geometry and its Nodal Configuration



Positive Element Boundary Traversal Convention



The area of the triangle is denoted by A and is given by

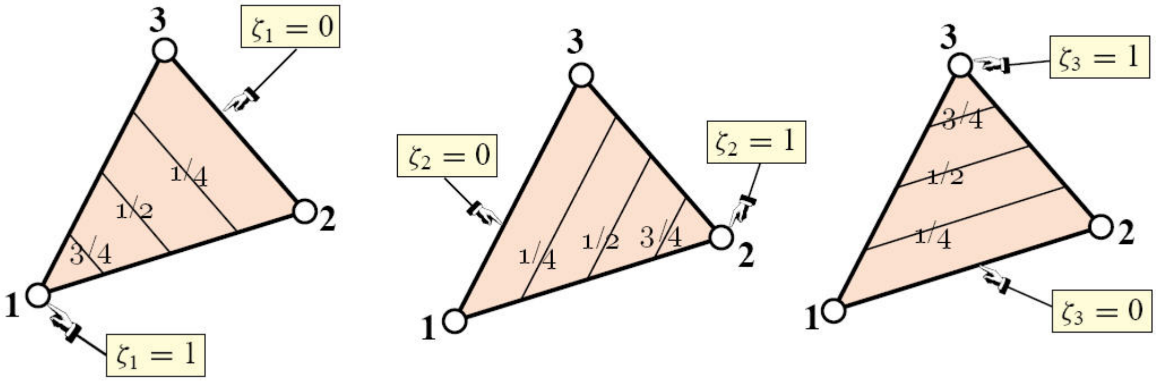
$$2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1).$$

001 USO SISTEMA COORDENADAS PARAMETRICO – COORDENADAS TRIANGULARES (CT) CURSO 2004-5

Introduction to FEM

Triangular Coordinates

ζ_1 ζ_2 ζ_3



$$\zeta_1 + \zeta_2 + \zeta_3 = 1$$

001 NOMBRES CON LOS QUE SE DENOMINAN A ESAS COORDENADAS EN LA “LITERATURA” CURSO 2004-5

Table 15.1 Names of element parametric coordinates in the FEM literature

Name	*Applicable to*
natural coordinates	all elements
isoparametric coordinates	isoparametric elements
shape function coordinates	isoparametric elements
barycentric coordinates	triangles, tetrahedra
Möbius coordinates	triangles
triangular coordinates	all triangles
area coordinates	straight-sided triangles

001 ECUACIONES EN COOR. TRIANGULARES” CURSO 2004-5

Equations

$$\zeta_i = constant \qquad (15.5)$$

represent a set of straight lines parallel to the side opposite to the i^{th} corner, as depicted in Figure 15.2. The equation of sides 1–2, 2–3 and 3–1 are $\zeta_1 = 0$, $\zeta_2 = 0$ and $\zeta_3 = 0$, respectively. The three corners have coordinates (1,0,0), (0,1,0) and (0,0,1). The three midpoints of the sides have coordinates $(\frac{1}{2}, \frac{1}{2}, 0)$, $(0, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2})$, the centroid has coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and so on. The coordinates are not independent because their sum is unity:

$$\zeta_1 + \zeta_2 + \zeta_3 = 1. \qquad (15.6)$$

001 UTILIZACION DE CT PARA FORMULAR UNA INTERPOLACION LINEAL CURSO 2004-5

Introduction to FEM

Using Triangular Coordinates to Formulate Linear Interpolation

Cartesian form

$$f(x,y) = a_0 + a_1 x + a_2 y$$

Variation defined by 3 corner values $f_1,\ f_2,\ f_3$

Natural form

$$f(\zeta_1,\zeta_2,\zeta_3) = f_1\zeta_1 + f_2\zeta_2 + f_3\zeta_3 = [\, f_1 \quad f_2 \quad f_3 \,] \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

$$= [\, \zeta_1 \quad \zeta_2 \quad \zeta_3 \,] \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

001 TRANSFORMACION DE COORDENADAS TRIANGULARES - CARTESIANAS CURSO 2004-5

Introduction to FEM

Relating Triangular & Cartesian Coordinates

Triangular to Cartesian:

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

Cartesian to Triangular:

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} 2A_{23} & y_{23} & x_{32} \\ 2A_{31} & y_{31} & x_{13} \\ 2A_{12} & y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

Here $x_{jk} = x_j - x_k \qquad y_{jk} = y_j - y_k$

A_{jk} denotes the area subtended by corners j, k and the origin of the x-y system

001 **DERIVADAS PARCIALES** CURSO 2004-5

Introduction to FEM

Partial Derivatives for Cartesian-Triangular Coordinate Transformations

$$\frac{\partial x}{\partial \zeta_i} = x_i \qquad \frac{\partial y}{\partial \zeta_i} = y_i$$

$$2A\frac{\partial \zeta_i}{\partial x} = y_{jk} \qquad 2A\frac{\partial \zeta_i}{\partial y} = x_{kj}$$

$$i = 1,2,3 \quad j = 2,3,1 \quad k = 3,1,2$$

001 **DERIVADAS PARCIALES CARTESIANAS DE UNA FUNCION EN CT** CURSO 2004-5

Introduction to FEM

Cartesian Partial Derivatives of a General Triangular-Coordinates Function

$$f = f(\zeta_1, \zeta_2, \zeta_3)$$

$$\frac{\partial f}{\partial x} = \frac{1}{2A}\left(\frac{\partial f}{\partial \zeta_1}y_{23} + \frac{\partial f}{\partial \zeta_2}y_{31} + \frac{\partial f}{\partial \zeta_3}y_{12}\right)$$
$$\frac{\partial f}{\partial y} = \frac{1}{2A}\left(\frac{\partial f}{\partial \zeta_1}x_{32} + \frac{\partial f}{\partial \zeta_2}x_{13} + \frac{\partial f}{\partial \zeta_3}x_{21}\right)$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}\begin{bmatrix} \frac{\partial f}{\partial \zeta_1} \\ \frac{\partial f}{\partial \zeta_2} \\ \frac{\partial f}{\partial \zeta_3} \end{bmatrix}$$

001 **TODO LISTO PARA FORMULAR EL ETL** CURSO 2004-5

With these mathematical ingredients in place we are now in a position to handle the derivation of straight-sided triangular elements, and in particular the linear triangle.

001 FORMULACION ETL CURSO 2004-5

FUNCIONES DE FORMA = CT

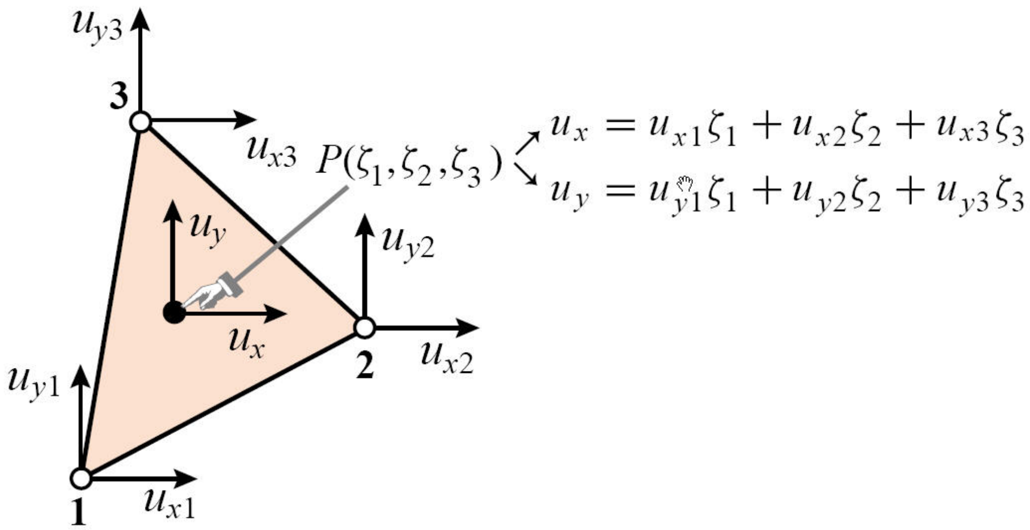
$$N_i^{(e)} = \zeta_i \text{ for } i = 1, 2, 3.$$

001 FORMULACION ETL CURSO 2004-5

INTERPOLACION DE LOS DESPLAZAMIENTOS”

Introduction to FEM

Displacement Interpolation over Triangle



$$u_x = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3$$
$$u_y = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3$$

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{N}\,\mathbf{u}^{(e)}$$

The shape functions are $N_i = \zeta_i\,,\quad i = 1,2,3$

OBTENCION DE LAS DEFORMACIONES

Differentiate the displacement interpolation wrt x,y **to get the strain-displacement relation**

$$\mathbf{e}(x,y) = \begin{bmatrix} \frac{\partial N_1^{(e)}}{\partial x} & 0 & \frac{\partial N_2^{(e)}}{\partial x} & 0 & \dots & \frac{\partial N_n^{(e)}}{\partial x} & 0 \\ 0 & \frac{\partial N_1^{(e)}}{\partial y} & 0 & \frac{\partial N_2^{(e)}}{\partial y} & \dots & 0 & \frac{\partial N_n^{(e)}}{\partial y} \\ \frac{\partial N_1^{(e)}}{\partial y} & \frac{\partial N_1^{(e)}}{\partial x} & \frac{\partial N_2^{(e)}}{\partial y} & \frac{\partial N_2^{(e)}}{\partial x} & \dots & \frac{\partial N_n^{(e)}}{\partial y} & \frac{\partial N_n^{(e)}}{\partial x} \end{bmatrix} \mathbf{u}^{(e)} = \mathbf{B}\mathbf{u}^{(e)}$$

B is called the strain-displacement matrix

$$\mathbf{B} = \mathbf{D}\mathbf{N}^{(e)}$$

Strain-Displacement Relations

$$\mathbf{e} = \mathbf{D}\mathbf{N}\,\mathbf{u}^{(e)} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{B}\mathbf{u}^{(e)}$$

Note that the strains are *constant* over the element. This is the origin of the name *constant strain triangle* (CST) given it in many finite element publications.

001 FORMULACION ETL CURSO 2004-5
OBTENCION DE LAS TENSIONES

$$\boldsymbol{\sigma} = \mathbf{E}\,\mathbf{e} = \mathbf{E}\mathbf{B}\mathbf{u}^{(e)}$$

Stress-Strain Relations

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{E}\mathbf{e}$$

Because the strains are constant, so are the stresses.

Element stiffness matrix

$$\mathbf{K}^{(e)} = \int_{\Omega^{(e)}} h\, \mathbf{B}^T \mathbf{E} \mathbf{B}\, d\Omega^{(e)}$$

Because **B** and **E** are constant over the triangle area:

$$\mathbf{K}^{(e)} = \mathbf{B}^T \mathbf{E} \mathbf{B} \int_{\Omega^{(e)}} h\, d\Omega^{(e)}$$

$$= \frac{1}{4A^2} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \int_{\Omega^{(e)}} h\, d\Omega^{(e)}$$

Introduction to FEM

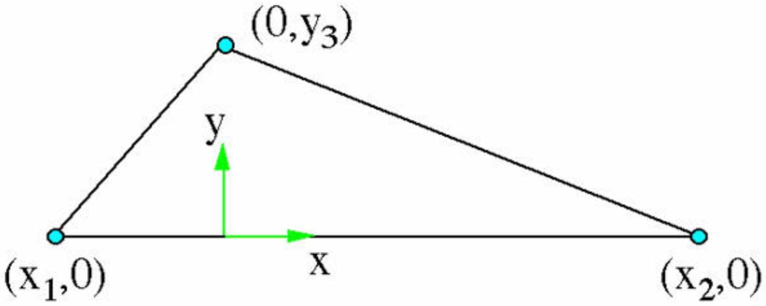
If the Plate Thickness h is Constant

$$\mathbf{K}^{(e)} = \frac{h}{4A} \begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

MATRIZ DE RIGIDEZ DEL ELEMENTO TRIANGULAR DE TURNER …

The birth of practical FEA: Turner, Clough, Martin and Topp, "Stiffness and Deflection Analysis of Complex Structures," Journal of the Aeronautical Sciences, Sept. 1956.

Turner, et. al. originally used an element coordinate system:



Use the shorthand notation $x_{21} = x_2 - x_1$.

In these coordinates, [k] is:

$$[k]= \frac{Eh}{2(1-\nu^2)}\begin{bmatrix} \frac{y_3}{x_{21}} & \frac{\nu x_2}{x_{21}} & -\frac{y_3}{x_{21}} & -\frac{\nu x_1}{x_{21}} & 0 & -\nu \\ \frac{\nu x_2}{x_{21}} & \frac{x_2^2}{y_3 x_{21}} & -\frac{\nu x_2}{x_{21}} & -\frac{x_1 x_2}{y_3 x_{21}} & 0 & -\frac{x_2}{y_3} \\ -\frac{y_3}{x_{21}} & -\frac{\nu x_2}{x_{21}} & \frac{y_3}{x_{21}} & \frac{\nu x_1}{x_{21}} & 0 & \nu \\ -\frac{\nu x_1}{x_{21}} & -\frac{x_1 x_2}{y_3 x_{21}} & \frac{\nu x_1}{x_{21}} & \frac{x_1^2}{y_3 x_{21}} & 0 & \frac{x_1}{y_3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\nu & -\frac{x_2}{y_3} & \nu & \frac{x_1}{y_3} & 0 & \frac{x_{21}}{y_3} \end{bmatrix} + \frac{Gh}{2}\begin{bmatrix} \frac{x_2^2}{y_3 x_{21}} & \frac{x_2}{x_{21}} & -\frac{x_1 x_2}{y_3 x_{21}} & -\frac{x_2}{x_{21}} & -\frac{x_2}{y_3} & 0 \\ \frac{x_2}{x_{21}} & \frac{y_3}{x_{21}} & -\frac{x_1}{x_{21}} & -\frac{y_3}{x_{21}} & -1 & 0 \\ -\frac{x_1 x_2}{y_3 x_{21}} & -\frac{x_1}{x_{21}} & \frac{x_1^2}{y_3 x_{21}} & \frac{x_1}{x_{21}} & \frac{x_1}{y_3} & 0 \\ -\frac{x_2}{x_{21}} & -\frac{y_3}{x_{21}} & \frac{x_1}{x_{21}} & \frac{y_3}{x_{21}} & 1 & 0 \\ -\frac{x_2}{y_3} & -1 & \frac{x_1}{y_3} & 1 & \frac{x_{21}}{y_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (19)$$

Consistent node force vector

$$\mathbf{f}^{(e)} = \int_{\Omega^{(e)}} h\,\mathbf{N}^T \mathbf{b}\, d\Omega^{(e)} + \int_{\Gamma^{(e)}} h\,\mathbf{N}^T \hat{\mathbf{t}}\, d\Gamma^{(e)}$$

body force **surface force**

Consistent Node Force Vector for Body Forces

$$\mathbf{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

$$\mathbf{f}^{(e)} = \int_{\Omega^{(e)}} h\,(\mathbf{N}^{(e)})^T \mathbf{b}\, d\Omega^{(e)} = \int_{\Omega^{(e)}} h \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_1 \\ \zeta_2 & 0 \\ 0 & \zeta_2 \\ \zeta_3 & 0 \\ 0 & \zeta_3 \end{bmatrix} \mathbf{b}\; d\Omega^{(e)}$$

If Body Forces are Constant over Element

Using *

$$\int_{\Omega^{(e)}} \zeta_1\; d\Omega^{(e)} = \int_{\Omega^{(e)}} \zeta_2\; d\Omega^{(e)} = \int_{\Omega^{(e)}} \zeta_3\; d\Omega^{(e)} = \tfrac{1}{3} A$$

we get

$$\mathbf{f}^{(e)} = \frac{Ah}{3} \begin{bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{bmatrix}$$

same result as straightforward "load lumping"

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* Above integrals are instances of general formula

$$\frac{1}{2A} \int_{\Omega^{(e)}} \zeta_1^i\, \zeta_2^j\, \zeta_3^k\; d\Omega^{(e)} = \frac{i!\, j!\, k!}{(i+j+k+2)!}, \quad i \ge 0,\ j \ge 0,\ k \ge 0$$

valid for triangles with *straight* sides

001 **EJERCICIO 1** CURSO 2004-5

EXERCISE 15.1

[A:15] Assume that the 3-node plane stress triangle has *variable* thickness defined over the element by the linear interpolation formula

$$h(\zeta_1, \zeta_2, \zeta_3) = h_1\zeta_1 + h_2\zeta_2 + h_3\zeta_3, \tag{E15.1}$$

where h_1, h_2 and h_3 are the thicknesses at the corner nodes. Show that the element stiffness matrix is still given by (15.22) but with h replaced by the mean thickness $h_m = (h_1 + h_2 + h_3)/3$. *Hint*: use (15.21) and (15.27).

$$\mathbf{K}^{(e)} = Ah\,\mathbf{B}^T\mathbf{E}\mathbf{B} = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}. \tag{15.22}$$

$$\mathbf{K}^{(e)} = \mathbf{B}^T\mathbf{E}\mathbf{B}\int_{\Omega^{(e)}} h\,d\Omega^{(e)}$$

$$= \frac{1}{4A^2}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \int_{\Omega^{(e)}} h\,d\Omega^{(e)}. \tag{15.21}$$

$$\frac{1}{2A}\int_{\Omega^{(e)}} \zeta_1^i\,\zeta_2^j\,\zeta_3^k\,d\Omega^{(e)} = \frac{i!\,j!\,k!}{(i+j+k+2)!}, \quad i \ge 0,\ j \ge 0,\ k \ge 0. \tag{15.27}$$

001 **EJERCICIO 2** CURSO 2004-5

EXERCISE 15.2

[A:20] The exact integrals of triangle-coordinate monomials over a straight-sided triangle are given by the formula

$$\frac{1}{2A}\int_A \zeta_1^i\,\zeta_2^j\,\zeta_3^k\,dA = \frac{i!\,j!\,k!}{(i+j+k+2)!} \tag{E15.2}$$

where A denotes the area of the triangle, and i, j and k are nonnegative integers. Tabulate the right-hand side for combinations of exponents i, j and k such that $i + j + k \le 3$, beginning with $i = j = k = 0$. Remember that $0! = 1$. (*Labor-saving hint*: don't bother repeating exponent permutations; for example $i = 2, j = 1, k = 0$ and $i = 1, j = 2, k = 0$ are permutations of the same thing. Hence one needs to tabulate only cases in which $i \ge j \ge k$).

001 **EJERCICIO 3** CURSO 2004-5

EXERCISE 15.3

[A/C:20] Compute the consistent node force vector $\mathbf{f}^{(e)}$ for body loads over a linear triangle, if the element thickness varies as per (E15.1), $b_x = 0$, and $b_y = b_{y1}\zeta_1 + b_{y2}\zeta_2 + b_{y3}\zeta_3$. Check that for $h_1 = h_2 = h_3 = h$ and $b_{y1} = b_{y2} = b_{y3} = b_y$ you recover (15.26). For the integrals over the triangle area use the formula (E15.2).

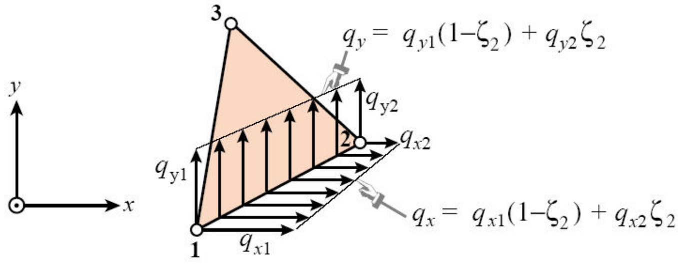
Partial result: $f_{y1} = (A/60)[b_{y1}(6h_1 + 2h_2 + 2h_3) + b_{y2}(2h_1 + 2h_2 + h_3) + b_{y3}(2h_1 + h_2 + 2h_3)]$.

EXERCISE 15.4

[A/C:20] Derive the formula for the consistent force vector $\mathbf{f}^{(e)}$ of a linear triangle of constant thickness h, if side 1–2 ($\zeta_3 = 0$, $\zeta_2 = 1 - \zeta_1$), is subject to a linearly varying boundary force $\mathbf{q} = h\hat{\mathbf{t}}$ such that

$$q_x = q_{x1}\zeta_1 + q_{x2}\zeta_2 = q_{x1}(1 - \zeta_2) + q_{x2}\zeta_2, \quad q_y = q_{y1}\zeta_1 + q_{y2}\zeta_2 = q_{y1}(1 - \zeta_2) + q_{y2}\zeta_2. \tag{E15.3}$$

This "line force" $\mathbf{q}$ has dimension of force per unit of side length.



Procedure. Use the last term of the line integral (14.21), in which $\hat{\mathbf{t}}$ is replaced by $\mathbf{q}/h$, and show that since the contribution of sides 2-3 and 3-1 to the line integral vanish,

$$W^{(e)} = \int_{\Omega^{(e)}} h\,\mathbf{u}^T\mathbf{b}\,d\Omega^{(e)} + \int_{\Gamma^{(e)}} h\,\mathbf{u}^T\,\hat{\mathbf{t}}\,d\Gamma^{(e)} \tag{14.21}$$

$$W^{(e)} = \left(\mathbf{u}^{(e)}\right)^T \mathbf{f}^{(e)} = \int_{\Gamma^{(e)}} \mathbf{u}^T\mathbf{q}\,d\Gamma^{(e)} = \int_0^1 \mathbf{u}^T\mathbf{q}\,L_{21}\,d\zeta_2, \tag{E15.4}$$

where L_{21} is the length of side 1-2. Replace $u_x(\zeta_2) = u_{x1}(1-\zeta_2) + u_{x2}\zeta_2$; likewise for u_y, q_x and q_y, integrate and identify with the inner product shown as the second term in (E15.4). Partial result: $f_{x1} = L_{21}(2q_{x1}+q_{x2})/6$, $f_{x3} = f_{y3} = 0$. *Note*. The following *Mathematica* script solves this Exercise. If you decide to use it, explain the logic.

```
ClearAll[ux1,uy1,ux2,uy2,ux3,uy3,z2,L12];
ux=ux1*(1-z2)+ux2*z2; uy=uy1*(1-z2)+uy2*z2;
qx=qx1*(1-z2)+qx2*z2; qy=qy1*(1-z2)+qy2*z2;
We=Simplify[L12*Integrate[qx*ux+qy*uy,{z2,0,1}]];
fe=Table[Coefficient[We,{ux1,uy1,ux2,uy2,ux3,uy3}[[i]]],{i,1,6}];
fe=Simplify[fe]; Print["fe=",fe];
```

001 **EJERCICIO 5** **CURSO 2004-5**

[C+N:15] Compute the entries of $\mathbf{K}^{(e)}$ for the following plane stress triangle:

$$x_1 = 0,\ y_1 = 0,\ x_2 = 3,\ y_2 = 1,\ x_3 = 2,\ y_3 = 2,$$
$$\mathbf{E} = \begin{bmatrix} 100 & 25 & 0 \\ 25 & 100 & 0 \\ 0 & 0 & 50 \end{bmatrix}, \qquad h = 1. \tag{E15.5}$$

This may be done by hand (it is a good exercise in matrix multiplication) or (more quickly) using the following *Mathematica* script:

```
Stiffness3NodePlaneStressTriangle[{{x1_,y1_},{x2_,y2_},{x3_,y3_}},
 Emat_,{h_}]:=Module[{x21,x13,x32,y12,y31,y23,A,Be,Ke},
   A=Simplify[(x2*y3-x3*y2+(x3*y1-x1*y3)+(x1*y2-x2*y1))/2];
   {x21,x13,x32}={x2-x1,x1-x3,x3-x2};
   {y12,y31,y23}={y1-y2,y3-y1,y2-y3};
   Be={{y23,0,y31,0,y12,0},{0,x32,0,x13,0,x21},
       {x32,y23,x13,y31,x21,y12}}/(2*A);
   Ke=A*h*Transpose[Be].Emat.Be;Return[Ke]];

Ke=Stiffness3NodePlaneStressTriangle[{{0,0},{3,1},{2,2}},
      {{100,25,0},{25,100,0},{0,0,50}},{1}];
Print["Ke=",Ke//MatrixForm];
Print["eigs of Ke=",Chop[Eigenvalues[N[Ke]]]];
Show[Graphics[Line[{{0,0},{3,1},{2,2},{0,0}}]],Axes->True];
```

Check it out: $K_{11} = 18.75$, $K_{66} = 118.75$. The last statement draws the triangle.

001 **EJERCICIO 6** **CURSO 2004-5**

EXERCISE 15.6

[A+C:15] Show that the sum of the rows (and columns) 1, 3 and 5 of $\mathbf{K}^{(e)}$ as well as the sum of rows (and columns) 2, 4 and 6 must vanish. Check it with the foregoing script.

001 **EJERCICIO 7** **CURSO 2004-5**

EXERCISE 15.7

[A/C:30] Let $p(\zeta_1, \zeta_2, \zeta_3)$ represent a *polynomial* expression in the natural coordinates. The integral

$$\int_{\Omega^{(e)}} p(\zeta_1, \zeta_2, \zeta_3)\, d\Omega \tag{E15.6}$$

over a straight-sided triangle can be computed symbolically by the following *Mathematica* module:

```
IntegrateOverTriangle[expr_,tcoord_,A_,max_]:=Module [{p,i,j,k,z1,z2,z3,c,s=0},
   p=Expand[expr];  {z1,z2,z3}=tcoord;
   For [i=0,i<=max,i++, For [j=0,j<=max,j++, For [k=0,k<=max,k++,
      c=Coefficient[Coefficient[Coefficient[p,z1,i],z2,j],z3,k];
      s+=2*c*(i!*j!*k!)/((i+j+k+2)!);
      ]]];
   Return[Simplify[A*s]] ];
```

This is referenced as `int=IntegrateOverTriangle[p,{z1,z2,z3},A,max]`. Here `p` is the polynomial to be integrated, `z1`, `z2` and `z3` denote the symbols used for the triangular coordinates, `A` is the triangle area and `max` the highest exponent appearing in a triangular coordinate. The module name returns the integral. For example, if `p=16+5*b*z2^2+z1^3+z2*z3*(z2+z3)` the call `int=IntegrateOverTriangle[p,{z1,z2,z3},A,3]` returns `int=A*(97+5*b)/6`. Explain how the module works.

EXERCISE 15.8

[C+D:25] Access the file `Trig3PlaneStress.nb` from the course Web site by clicking on the appropriate link in Chapter 15 Index. This is a *Mathematica* 4.1 Notebook that does plane stress FEM analysis using the 3-node linear triangle.

Download the Notebook into your directory. Load into *Mathematica*. Execute the top 7 input cells (which are actually initialization cells) so the necessary modules are compiled. Each cell is preceded by a short comment cell which outlines the purpose of the modules it holds. Notes: (1) the plot-module cell may take a while to run through its tests; be patient; (2) to get rid of unsightly messages and silly beeps about similar names, initialize each cell twice.

After you are satisfied everything works fine, run the cantilever beam problem, which is defined in the last input cell.

After you get a feel of how this code operate, study the source. Prepare a hierarchical diagram of the modules,[4] beginning with the main program of the last cell. Note which calls what, and briefly explain the purpose of each module. Return this diagram as answer to the homework. You do not need to talk about the actual run and results; those will be discussed in Part III.

---

[4] A hierarchical diagram is a list of modules and their purposes, with indentation to show dependence, similar to the table of contents of a book. For example, if module `AAAA` calls `BBBB` and `CCCC`, and `BBBB` calld `DDDD`, the hierarchical diagram may look like:

```
AAAA -  purpose of AAAA
  BBBB  - purpose of BBBB
    DDDD - purpose of DDDD
  CCCC  - purpose of CCCC
```

Hint: a hierarchical diagram for `Trig3PlaneStress.nb` begins like

```
Main program in Cell 8 - drives the FEM analysis
   GenerateNodes - generates node coordinates of regular mesh
   GenerateTriangles - generate element node lists of regular mesh
   ........
```