5.2. Triángulos de Tres Nodos para Tensión Plana – Mathematica.

Partiendo de la lectura del capítulo 15 del Prof. Carlos A. Felippa, en esta sección elaboramos un documento de "Mathematica" definiendo el elemento que se presenta en este capítulo, utilizándolo para obtener la Matriz de Rigidez tal y como la presento Turner.

CHAPTER 15. Triángulos de Tres Nodos para Tensión Plana.
Carlos A. Felippa.

ELEMENTO TRIANGULAR LINEAR - 3 NODOS
Elemento Triangular Lineal Deformación Constante
v.2012

INICIO.

- **Directorio de Trabajo e Inicio.**

In[64]:=
```
SetDirectory["G:\\TRIANGULO_03"]
```
Out[64]=
```
G:\TRIANGULO_03
```

```
Off[General::spell1]
Off[General::spell]
```

- **Referencia.**

```
Lección = Import["001.jpg"];
Show[Lección, ImageSize → 300]
```

15
The Linear Plane Stress Triangle

In[65]:=
```
Show[Import["0001.jpg"], ImageSize → 1000]
```
Out[65]=

§15.1. Introduction

This Chapter derives element stiffness equations of three-node triangles constructed with linear displacements for the plane stress problem formulated in Chapter 14. These elements have six displacement degrees of freedom, which are placed at the *connection nodes*. There are two main versions that differ on where the connection nodes are located:

1. The *Turner triangle* has connection nodes located at the corners.
2. The *Veubeke equilibrium triangle* has connection nodes located at the side midpoints.

The triangle geometry is defined by the corner locations or *geometric nodes* in both cases. Of the two versions, the Turner triangle is by far the most practically important one in solid and structural mechanics.[1] Thus most of the material in this Chapter is devoted to it. It enjoys several important properties:

(i) It belongs to both the isoparametric and subparametric element families, which are introduced in the next Chapter.

(ii) It allows closed form derivations for the stiffness matrix and consistent force vector without need for numerical integration.

(iii) It cannot be improved by the addition of internal degrees of freedom.

Properties (ii) and (iii) are shared by the Veubeke equilibrium triangle. Since this model is rarely used in structural applications it is covered only as advanced material in §15.5.

In[66]:=

```
Show[Import["0002.jpg"], ImageSize → 1000]
```

Out[66]=

The Turner triangle is not a good performer for structural stress analysis. It is still used in problems that do not require high accuracy, as well as in non-structural applications such as thermal and electromagnetic analysis. One reason is that triangular meshes are easily generated over arbitrary two-dimensional domains using techniques such as Delaunay triangulation.

§15.2. Background

§15.2.1. Parametric Representation of Functions

The concept of *parametric representation* of functions is crucial in modern FEM. Together with multidimensional numerical integration, it is a key enabling tool for developing elements in two and three space dimensions.[2] Without these tools the developer would become lost in an algebraic maze as element geometry and shape functions get more complicated. The essentials of parametric representation can be illustrated through a simple example. Consider the following alternative representations of the unit-circle function, $x^2 + y^2 = 1$:

$$\text{(I)}\ \ y = \sqrt{1-x^2}, \qquad \text{(II)}\ \ x = \cos\theta \ \text{ and } y = \sin\theta. \tag{15.1}$$

The direct representation (I) fits the conventional function notation, i.e., $y = f(x)$. Given a value of x, it returns one or more y. On the other hand, the parametric representation (II) is indirect: both x

[1] The triangle was one of the two plane-stress continuum elements presented by Turner, Clough, Martin and Topp in their 1956 paper [323]. This publication is widely regarded as the start of the present FEM. The derivation was not done, however, with assumed displacements. See **Notes and Bibliography** at the end of this Chapter.

[2] Numerical integration is not useful for the triangular elements covered here, but essential in the more complicated iso-P models covered in Chapters 16ff.

In[67]:=

```
Show[Import["0003.jpg"], ImageSize → 1000]
```

Out[67]=

and y are given in terms of one parameter, the angle θ. Elimination of θ through the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ recovers $x^2 + y^2 = 1$. But there are situations in which working with the parametric form throughout the development is more convenient. Continuum finite elements provide a striking illustration of this point.

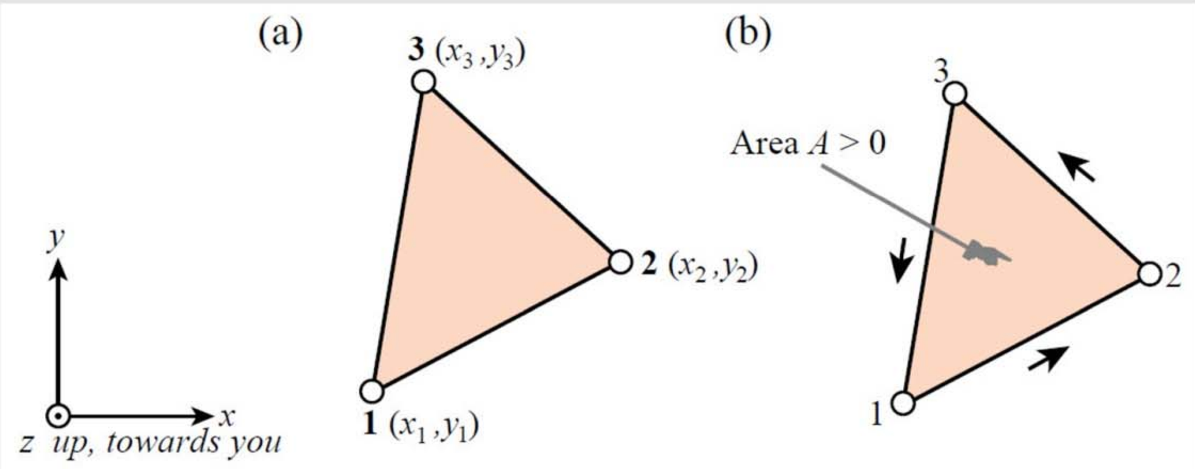
GEOMETRIA DEL TRIANGULO Y SISTEMAS DE COORDENADAS

- **Area.**

In[68]:=

```
Triangulo = Import["002.jpg"];
Show[Triangulo, ImageSize → 750]
```

Out[69]=



In[70]:=
```
Area = Import["003.jpg"];
Show[Area, ImageSize → 750]
```

Out[71]=

$$2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1). \qquad (15.3)$$

Definición Matriz T

```
T = (1   1   1
     x1  x2  x3
     y1  y2  y3)
```

```
{{1, 1, 1}, {x1, x2, x3}, {y1, y2, y3}}
```

Definición del Area del Elemento Triangular

```
AreaT[x1_, y1_, x2_, y2_, x3_, y3_] = Det[T] / 2;
```

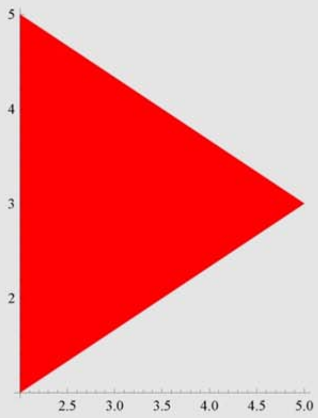
```
AreaT[x1, y1, x2, y2, x3, y3] // MatrixForm
```

$$\frac{1}{2}\,(-x2\,y1 + x3\,y1 + x1\,y2 - x3\,y2 - x1\,y3 + x2\,y3)$$

Ejemplo - $DIBUJO

```
ncoor = {{2, 5}, {2, 1}, {5, 3}};
```

```
Show[Graphics[RGBColor[1, 0, 0]], Graphics[AbsoluteThickness[2]], Graphics[Polygon[ncoor]], Axes → True]
```




```
AreaT[x1, y1, x2, y2, x3, y3] /. {x1 → 2, y1 → 5, x2 → 2, y2 → 1, x3 → 5, y3 → 3}
```

▫ Referencia

In[72]:=
```
Show[Import["0004.jpg"], ImageSize → 1000]
```

Out[72]=

§15.2.2. Geometry

The geometry of the 3-node triangle shown in Figure 15.1(a) is specified by the location of its three corner nodes on the $\{x, y\}$ plane. Nodes are labelled 1, 2, 3 while traversing the sides in *counterclockwise* fashion. Their location is defined by their Cartesian coordinates: $\{x_i,\ y_i\}$ for $i = 1, 2, 3$.

The Turner triangle has six degrees of freedom, defined by the six corner displacement components $\{u_{xi}, u_{yi}\}$, for $i = 1, 2, 3$. The interpolation of the internal displacements $\{u_x, u_y\}$ from these six values is studied in §15.3, after triangular coordinates are introduced. The triangle area can be obtained as

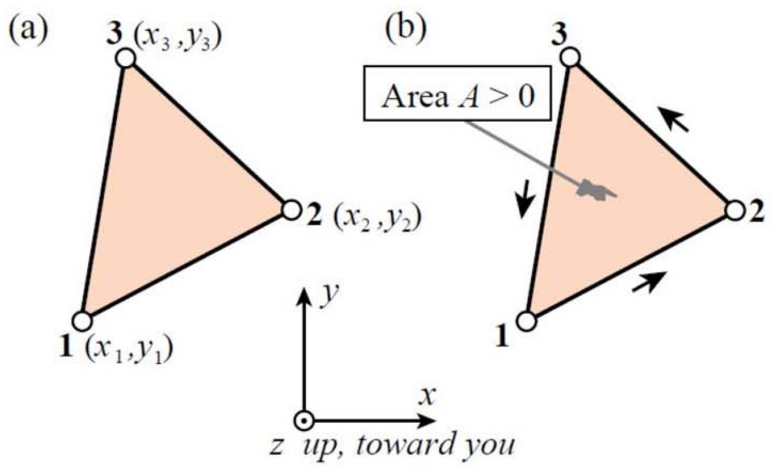


FIGURE 15.1. The three-node, linear-displacement plane stress triangular element: (a) geometry; (b) area and positive boundary traversal.

$$2A = \det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1). \qquad (15.2)$$

The area given by 15.2 is a *signed* quantity. It is positive if the corners are numbered in cyclic counterclockwise order (when looking down from the $+z$ axis), as illustrated in Figure 15.1(b). This convention is followed in the sequel.

▪ Coordenadas Triangulares.

In[73]:=
```
Coordenadas = Import["005.jpg"];
Show[Coordenadas, ImageSize → 750]
```

Out[74]=

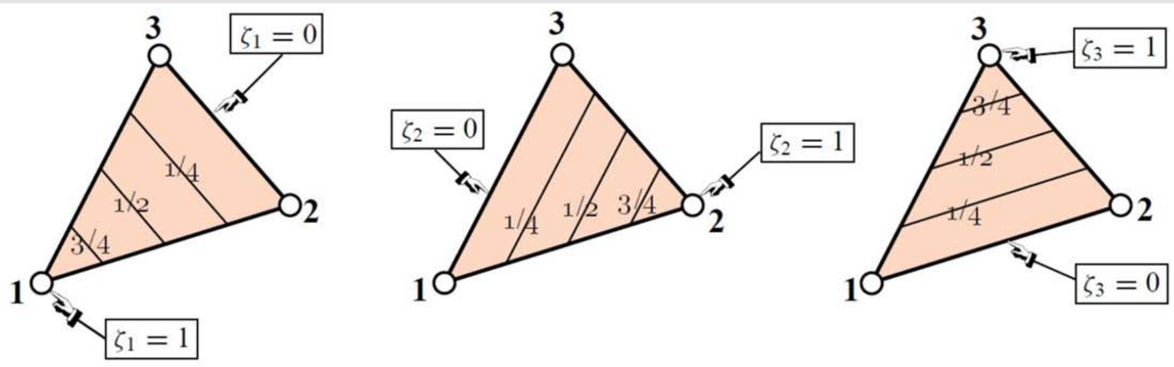


Figure 15.2. Triangular coordinates.

In[75]:=
```
Relación = Import["006.jpg"];
Show[Relación, ImageSize → 750]
```

Out[76]=

$$\zeta_1 + \zeta_2 + \zeta_3 = 1. \qquad (15.6)$$

▫ Referencia

In[77]:=
```
Show[Import["0005.jpg"], ImageSize → 1000]
```

Out[77]=

§15.2.3. Triangular Coordinates

Points of the triangle may also be located in terms of a *parametric* coordinate system:

$$\zeta_1,\ \zeta_2,\ \zeta_3. \quad (15.3)$$

In the literature these 3 parameters receive an astonishing number of names, as the list collected in Table 15.1 shows. In the sequel the name *triangular coordinates* will be used to emphasize the close association with this particular geometry.

Equations

$$\zeta_i = constant \quad (15.4)$$

represent a set of straight lines parallel to the side opposite to the i^{th} corner, as depicted in Figure 15.2. The equations of sides 2–3, 3–1 and 1–2 are $\zeta_1 = 0$, $\zeta_2 = 0$ and $\zeta_3 = 0$, respectively. The three corners have coordinates (1,0,0), (0,1,0) and (0,0,1). The three midpoints of the sides have coordinates $(\frac{1}{2}, \frac{1}{2}, 0)$, $(0, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2})$, the centroid has coordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and so on. The coordinates are not independent because their sum is unity:

$$\zeta_1 + \zeta_2 + \zeta_3 = 1. \quad (15.5)$$

In[78]:=
```
Show[Import["0006.jpg"], ImageSize → 1000]
```

Out[78]=

Table 15.1 Names of element parametric coordinates

Name	*Applicable to*
natural coordinates	all elements
isoparametric coordinates	isoparametric elements
shape function coordinates	isoparametric elements
barycentric coordinates	simplices (triangles, tetrahedra, ...)
Möbius coordinates	triangles
triangular coordinates	all triangles
area (also written "areal") coordinates	straight-sided triangles

Triangular coordinates normalized as per $\zeta_1 + \zeta_2 + \zeta_3 = 1$ are often qualified as "homogeneous" in the mathematical literature.

Remark 15.1. In pre-1970 FEM publications, triangular coordinates were often called *area coordinates*, and occasionally *areal coordinates*. This comes from the following interpretation: $\zeta_i = A_{jk}/A$, where A_{jk} is the area subtended by the subtriangle formed by the point P and corners j and k, in which j and k are 3-cyclic permutations of i. Historically this was the way coordinates were defined in 1960s papers. However this relation does not carry over to general isoparametric triangles with curved sides and thus it is not used here.

Interpolación Lineal.

Referencia.

In[79]:=
```
Show[Import["0007.jpg"], ImageSize → 1000]
```

Out[79]=

§15.2.4. Linear Interpolation

Consider a function $f(x, y)$ that varies *linearly* over the triangle domain. In terms of Cartesian coordinates it may be expressed as

$$f(x, y) = a_0 + a_1 x + a_2 y, \tag{15.6}$$

where a_0, a_1 and a_2 are coefficients to be determined from three conditions. In finite element work such conditions are often the *nodal values* taken by f at the corners:

$$f_1, \; f_2, \; f_3. \tag{15.7}$$

The expression in triangular coordinates makes direct use of those three values:

$$f(\zeta_1, \zeta_2, \zeta_3) = f_1\zeta_1 + f_2\zeta_2 + f_3\zeta_3 = [\, f_1 \;\; f_2 \;\; f_3 \,] \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = [\, \zeta_1 \;\; \zeta_2 \;\; \zeta_3 \,] \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}. \tag{15.8}$$

Formula 15.8 is called a *linear interpolant* for f.

Transformación de Coordenadas - Triangulares a Cartesianas.

In[80]:=
```
TriangularesACartesianas = Import["007.jpg"];
Show[TriangularesACartesianas, ImageSize → 750]
```

Out[81]=

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}. \tag{15.10}$$

Definición Vector Coordenadas Triangulares

$$Ct = \begin{pmatrix} \xi 1 \\ \xi 2 \\ \xi 3 \end{pmatrix};$$

Definición Funcion Transformación de Triangulares a Cartesianas

```
TaC[ξ1_, ξ2_, ξ3_] = T.Ct;
```

```
TaC[ξ1, ξ2, ξ3] // MatrixForm
```

$$\begin{pmatrix} \xi 1 + \xi 2 + \xi 3 \\ x1\,\xi 1 + x2\,\xi 2 + x3\,\xi 3 \\ y1\,\xi 1 + y2\,\xi 2 + y3\,\xi 3 \end{pmatrix}$$

Ejemplo - $DIBUJO

```
TaC[1/4, 1/2, 1/4] /. {x1 → 2, y1 → 5, x2 → 2, y2 → 1, x3 → 5, y3 → 3} // MatrixForm
```

$$\begin{pmatrix} 1 \\ \frac{11}{4} \\ \frac{5}{2} \end{pmatrix}$$

▫ Referencia

In[82]:=
```
Show[Import["0008.jpg"], ImageSize → 1000]
```

Out[82]=

§15.2.5. Coordinate Transformations

Quantities that are closely linked with the element geometry are best expressed in triangular coordinates. On the other hand, quantities such as displacements, strains and stresses are usually expressed in the Cartesian system $\{x, y\}$. Thus we need transformation equations through which it is possible to pass from one coordinate system to the other.

Cartesian and triangular coordinates are linked by the relation

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}. \qquad (15.9)$$

The first equation says that the sum of the three coordinates is one. The next two express x and y linearly as homogeneous forms in the triangular coordinates. These are obtained by applying the linear interpolant 15.8 to the Cartesian coordinates: $x = x_1\zeta_1 + x_2\zeta_2 + x_3\zeta_3$ and $y = y_1\zeta_1 + y_2\zeta_2 + y_3\zeta_3$. Assuming $A \neq 0$, inversion of 15.9 yields

■ Transformación de Coordenadas - Cartesianas a Triangulares.

In[83]:=
```
CartesianasATriangulares = Import["008.jpg"];
Show[CartesianasATriangulares, ImageSize → 750]
```

Out[84]=

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} 2A_{23} & y_{23} & x_{32} \\ 2A_{31} & y_{31} & x_{13} \\ 2A_{12} & y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}. \qquad (15.11)$$

```
?Inverse
```

Inverse[*m*] gives the inverse of a square matrix *m*. ≫

```
T // MatrixForm
```

$$\begin{pmatrix} 1 & 1 & 1 \\ x1 & x2 & x3 \\ y1 & y2 & y3 \end{pmatrix}$$

```
Inverse[T] // MatrixForm
```

$$\begin{pmatrix} \frac{-x3\,y2+x2\,y3}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} & \frac{y2-y3}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} & \frac{-x2+x3}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} \\ \frac{x3\,y1-x1\,y3}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} & \frac{-y1+y3}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} & \frac{x1-x3}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} \\ \frac{-x2\,y1+x1\,y2}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} & \frac{y1-y2}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} & \frac{-x1+x2}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} \end{pmatrix}$$

```
Inverse[T] * 2 * AreaT[x1, y1, x2, y2, x3, y3] // MatrixForm
```

$$\begin{pmatrix} -x3\,y2+x2\,y3 & y2-y3 & -x2+x3 \\ x3\,y1-x1\,y3 & -y1+y3 & x1-x3 \\ -x2\,y1+x1\,y2 & y1-y2 & -x1+x2 \end{pmatrix}$$

Definición Vector Coordenadas Cartesianas

$$Cc = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix};$$

Definición Funcion Transformación de Cartesianas a Triangulares

```
In[86]:= Show[CartesianasATriangulares, ImageSize → 750]
```

Out[86]=

$$\begin{bmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{bmatrix} = \frac{1}{2A}\begin{bmatrix}x_2y_3-x_3y_2 & y_2-y_3 & x_3-x_2\\ x_3y_1-x_1y_3 & y_3-y_1 & x_1-x_3\\ x_1y_2-x_2y_1 & y_1-y_2 & x_2-x_1\end{bmatrix}\begin{bmatrix}1\\ x\\ y\end{bmatrix} = \frac{1}{2A}\begin{bmatrix}2A_{23} & y_{23} & x_{32}\\ 2A_{31} & y_{31} & x_{13}\\ 2A_{12} & y_{12} & x_{21}\end{bmatrix}\begin{bmatrix}1\\ x\\ y\end{bmatrix}. \quad (15.11)$$

```
CaT[x_, y_] = Inverse[T].Cc;
```

```
CaT[x, y] // MatrixForm
```

$$\begin{pmatrix} \frac{(-x2+x3)\,y}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{x\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{-x3\,y2+x2\,y3}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}\\ \frac{(x1-x3)\,y}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{x\,(-y1+y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{x3\,y1-x1\,y3}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}\\ \frac{(-x1+x2)\,y}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{x\,(y1-y2)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{-x2\,y1+x1\,y2}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} \end{pmatrix}$$

Definición Funcion Transformación para cada Coordenada Triangular

```
ξ1[x_, y_] = CaT[x, y][[1, 1]];
ξ2[x_, y_] = CaT[x, y][[2, 1]];
ξ3[x_, y_] = CaT[x, y][[1, 1]];
```

Ejemplo -$DIBUJO

```
TaC[1/4, 1/2, 1/4] /. {x1 → 2, y1 → 5, x2 → 2, y2 → 1, x3 → 5, y3 → 3} // MatrixForm
```

$$\begin{pmatrix}1\\ \frac{11}{4}\\ \frac{5}{2}\end{pmatrix}$$

Referencia

```
In[87]:= Show[Import["0009.jpg"], ImageSize → 1000]
```

Out[87]=

The first equation says that the sum of the three coordinates is one. The next two express x and y linearly as homogeneous forms in the triangular coordinates. These are obtained by applying the linear interpolant 15.8 to the Cartesian coordinates: $x = x_1\zeta_1 + x_2\zeta_2 + x_3\zeta_3$ and $y = y_1\zeta_1 + y_2\zeta_2 + y_3\zeta_3$. Assuming $A \neq 0$, inversion of 15.9 yields

$$\begin{bmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{bmatrix} = \frac{1}{2A}\begin{bmatrix}x_2y_3-x_3y_2 & y_2-y_3 & x_3-x_2\\ x_3y_1-x_1y_3 & y_3-y_1 & x_1-x_3\\ x_1y_2-x_2y_1 & y_1-y_2 & x_2-x_1\end{bmatrix}\begin{bmatrix}1\\ x\\ y\end{bmatrix} = \frac{1}{2A}\begin{bmatrix}2A_{23} & y_{23} & x_{32}\\ 2A_{31} & y_{31} & x_{13}\\ 2A_{12} & y_{12} & x_{21}\end{bmatrix}\begin{bmatrix}1\\ x\\ y\end{bmatrix}. \quad (15.10)$$

Here $x_{jk} = x_j - x_k$, $y_{jk} = y_j - y_k$, A is the triangle area given by 15.2 and A_{jk} denotes the area subtended by corners j, k and the origin of the x–y system. If this origin is taken at the centroid of the triangle, $A_{23} = A_{31} = A_{12} = A/3$.

■ Derivadas Parciales.

▫ Referencia

In[88]:=
```
Show[Import["0010.jpg"], ImageSize → 1000]
```

Out[88]=

§15.2.6. Partial Derivatives

From equations 15.9 and 15.10 we immediately obtain the following relations between partial derivatives:

$$\frac{\partial x}{\partial \zeta_i} = x_i, \qquad \frac{\partial y}{\partial \zeta_i} = y_i, \tag{15.11}$$

$$2A\frac{\partial \zeta_i}{\partial x} = y_{jk}, \qquad 2A\frac{\partial \zeta_i}{\partial y} = x_{kj}. \tag{15.12}$$

In 15.12 j and k denote the *3-cyclic permutations* of i. For example, if $i = 2$, then $j = 3$ and $k = 1$. The derivatives of a function $f(\zeta_1, \zeta_2, \zeta_3)$ with respect to x or y follow immediately from 15.12 and application of the chain rule:

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{2A}\left(\frac{\partial f}{\partial \zeta_1}y_{23} + \frac{\partial f}{\partial \zeta_2}y_{31} + \frac{\partial f}{\partial \zeta_3}y_{12}\right) \\ \frac{\partial f}{\partial y} &= \frac{1}{2A}\left(\frac{\partial f}{\partial \zeta_1}x_{32} + \frac{\partial f}{\partial \zeta_2}x_{13} + \frac{\partial f}{\partial \zeta_3}x_{21}\right) \end{aligned}} \tag{15.13}$$

In[89]:=
```
Show[Import["0011.jpg"], ImageSize → 1000]
```

Out[89]=

which in matrix form is

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \zeta_1} \\ \frac{\partial f}{\partial \zeta_2} \\ \frac{\partial f}{\partial \zeta_3} \end{bmatrix}. \tag{15.14}$$

With these mathematical ingredients in place we are now in a position to handle the derivation of straight-sided triangular elements, and in particular the Turner and Veubeke triangles.

FORMULACION DE ELEMENTO

■ Funciones de Forma - Triangulo Lineal de 3 Nodos - Funciones de Interpolación

```
N1 = ξ1[x, y];
N2 = ξ2[x, y];
N3 = ξ3[x, y];
```

```
N1
```

$$\frac{(-x2 + x3)\ y}{-x2\ y1 + x3\ y1 + x1\ y2 - x3\ y2 - x1\ y3 + x2\ y3} + \frac{x\ (y2 - y3)}{-x2\ y1 + x3\ y1 + x1\ y2 - x3\ y2 - x1\ y3 + x2\ y3} + \frac{-x3\ y2 + x2\ y3}{-x2\ y1 + x3\ y1 + x1\ y2 - x3\ y2 - x1\ y3 + x2\ y3}$$

▫ Referencia

In[90]:=
```
Show[Import["0012.jpg"], ImageSize → 1000]
```

Out[90]=

§15.3. The Turner Triangle

The simplest triangular element for plane stress (and in general, for 2D problems of variational index $m = 1$) is the three-node triangle with *linear shape functions*, with degrees of freedom located at the corners. The shape functions are simply the triangular coordinates. That is, $N_i^e = \zeta_i$ for $i = 1, 2, 3$. When applied to the plane stress problem, this element is called the Turner triangle.

▪ Interpolación de los Desplazamientos

In[91]:=
```
InterpolaciónG = Import["009.jpg"];
Show[InterpolaciónG, ImageSize → 750]
```

Out[92]=

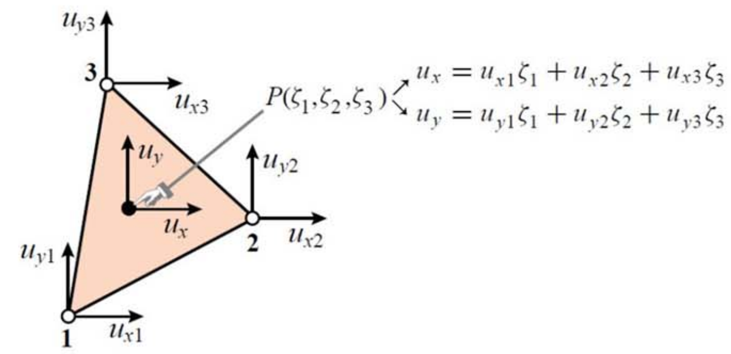


Figure 15.3. Displacement interpolation over triangle.

In[93]:=
```
InterpolaciónF = Import["010.jpg"];
Show[InterpolaciónF, ImageSize → 750]
```

Out[94]=

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{N}^{(e)}\mathbf{u}^{(e)}. \qquad (15.17)$$

In[95]:=
```
Show[Import["0013.jpg"], ImageSize → 1000]
```

Out[95]=

§15.3.1. Displacement Interpolation

For the plane stress problem we select the linear interpolation 15.8 for the displacement components u_x and u_y at an arbitrary point $P(\zeta_1, \zeta_2, \zeta_3)$:

$$u_x = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3, \qquad u_y = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3. \qquad (15.15)$$

The interpolation is illustrated in Figure 15.4. The two expressions in 15.15 can be combined in a matrix form that befits the expression (14.17) for an arbitrary plane stress element:

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{N}\,\mathbf{u}^e, \qquad (15.16)$$

where **N** is the matrix of shape functions.

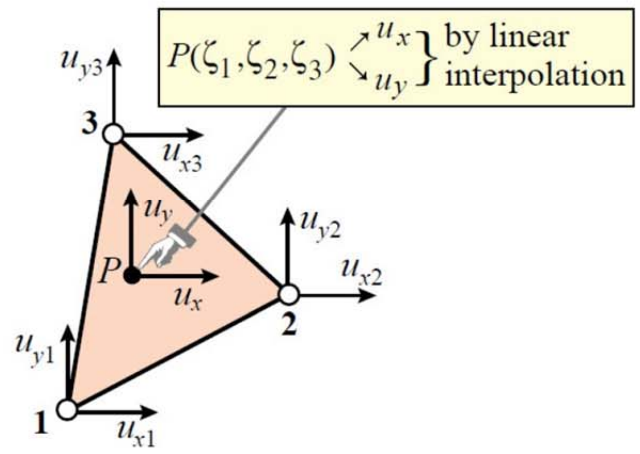


FIGURE 15.4. Displacement interpolation over triangle.

Interpolación valida para cualquier elemento con cualquier numero de nodos.

```
InterpolaciónGN = Import["011.jpg"];
Show[InterpolaciónGN, ImageSize → 750]
```

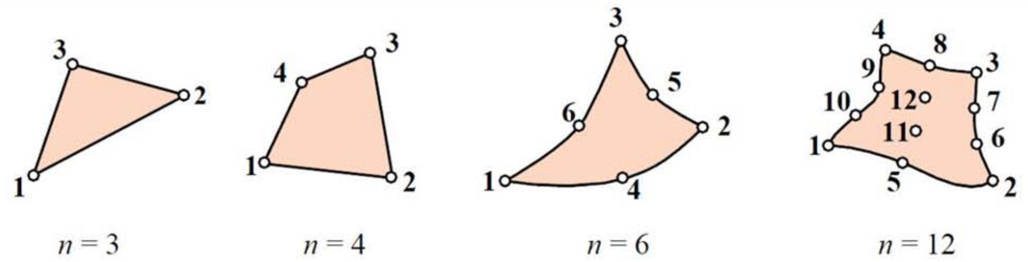


Figure 14.8. Example two-dimensional finite elements, characterized by their number of nodes n.

```
InterpolaciónFN = Import["012.jpg"];
Show[InterpolaciónFN, ImageSize → 750]
```

$$\mathbf{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} N_1^{(e)} & 0 & N_2^{(e)} & 0 & \dots & N_n^{(e)} & 0 \\ 0 & N_1^{(e)} & 0 & N_2^{(e)} & \dots & 0 & N_n^{(e)} \end{bmatrix} \mathbf{u}^{(e)} = \mathbf{N}^{(e)}\mathbf{u}^{(e)}. \quad (14.17)$$

```
Show[Import["0015.jpg"], ImageSize → 1000]
```

§14.4.1. Displacement Interpolation

The displacement field $\mathbf{u}^e(x,y)$ over the element is interpolated from the node displacements. We shall assume that the same interpolation functions are used for both displacement components.[3] Thus

$$u_x(x,y) = \sum_{i=1}^{n} N_i^e(x,y)\, u_{xi}, \qquad u_y(x,y) = \sum_{i=1}^{n} N_i^e(x,y)\, u_{yi}, \qquad (14.16)$$

where $N_i^e(x,y)$ are the element shape functions. In matrix form:

$$\mathbf{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & \dots & N_n^e & 0 \\ 0 & N_1^e & 0 & N_2^e & \dots & 0 & N_n^e \end{bmatrix} \mathbf{u}^e = \mathbf{N}\,\mathbf{u}^e. \qquad (14.17)$$

This $\mathbf{N}$ (with superscript e omitted to reduce clutter) is called the *shape function matrix*. It has dimensions $2 \times 2n$. For example, if the element has 4 nodes, $\mathbf{N}$ is 2×8.

The *interpolation condition* on the element shape function $N_i^e(x,y)$ states that it must take the value one at the i^{th} node and zero at all others. This ensures that the interpolation 14.17 is correct at the nodes. Additional requirements on the shape functions are stated in later Chapters.

[3] This is the so called *element isotropy* condition, which is studied and justified in advanced FEM courses.

******* Matriz Funciones de Forma - N**

```
Ne = ( N1  0  N2  0  N3  0
       0  N1  0  N2  0  N3 );
```

```
Ne // MatrixForm
```

$$\left(\frac{(-x2+x3)\,y}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{x\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{-x3\,y2+x2\,y3}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} \quad \dots \right.$$

$$\left. 0 \qquad \frac{(-x2+x3)\,y}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{\dots}{-x2\,y1+x \dots} \right.$$

Vector Desplazamientos Nodales - DATOS

```
DesplazamientosNodales = Import["013.jpg"];
Show[DesplazamientosNodales, ImageSize → 750]
```

$$\mathbf{u}^{(e)} = [\, u_{x1} \quad u_{y1} \quad u_{x2} \quad \dots \quad u_{xn} \quad u_{yn} \,]^T. \qquad (14.15)$$

```
ue = (ux1
      uy1
      ux2
      uy2
      ux3
      uy3);
```

```
Show[Import["0014.jpg"], ImageSize → 1000]
```

§14.4. Finite Element Equations

The necessary equations to apply the finite element method to the plane stress problem are collected here and expressed in matrix form. The domain of Figure 14.7(a) is discretized by a finite element mesh as illustrated in Figure 14.7(b). From this mesh we extract a generic element labeled e with $n \geq 3$ node points. In subsequent derivations the number n is kept *arbitrary*. Therefore, the formulation is applicable to arbitrary two-dimensional elements, for example those sketched in Figure 14.8.

To comfortably accommodate general element types, the node points will be labeled 1 through n. These are called *local node numbers*. Numbering will always start with corners.

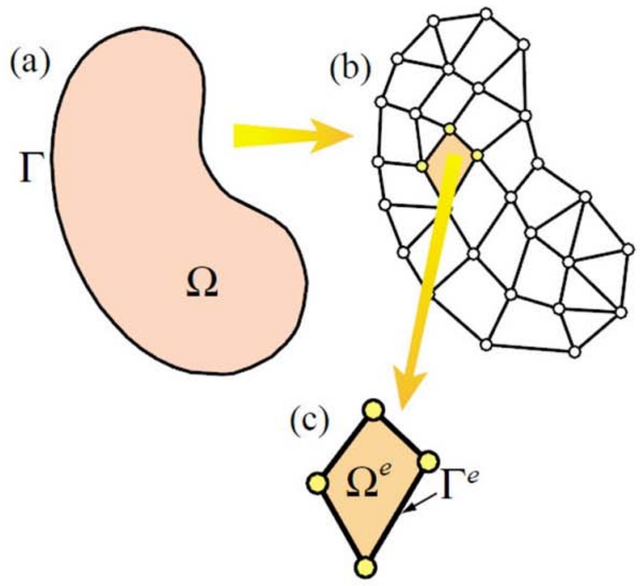


FIGURE 14.7. Finite element discretization and extraction of generic element.

The element domain and boundary are denoted by Ω^e and Γ^e, respectively. The element has $2n$ degrees of freedom. These are collected in the element node displacement vector in a node by node arrangement:

$$\mathbf{u}^e = [\, u_{x1} \quad u_{y1} \quad u_{x2} \quad \ldots \quad u_{xn} \quad u_{yn} \,]^T. \tag{14.15}$$

▫ Vector Desplazamientos en un Punto en función de los nodales

```
u[x_, y_] = Ne.ue;
```

```
u[x, y] // MatrixForm
```

```
( ux2 ( (x1-x3) y / (-x2 y1+x3 y1+x1 y2-x3 y2-x1 y3+x2 y3) + x (-y1+y3) / (-x2 y1+x3 y1+x1 y2-x3 y2-x1 y3+x2 y3) + (x3 y1-x1 y3) / (-x2 y1+x3 y1+x1 y2-x3 y2-x1 y3+x2 y3) ) + ux1 ( (-x2+x3) y / (-x2 y1+x3 y1+x1 y2-x3 y2-x1 y3+
  uy2 ( (x1-x3) y / (-x2 y1+x3 y1+x1 y2-x3 y2-x1 y3+x2 y3) + x (-y1+y3) / (-x2 y1+x3 y1+x1 y2-x3 y2-x1 y3+x2 y3) + (x3 y1-x1 y3) / (-x2 y1+x3 y1+x1 y2-x3 y2-x1 y3+x2 y3) ) + uy1 ( (-x2+x3) y / (-x2 y1+x3 y1+x1 y2-x3 y2-x1 y3+
```

```
Dimensions[u[x, y]]
```

```
{2, 1}
```

-:) dos filas por una columna

▫ Valor del Desplazamiento en x

```
u[x, y][[1, 1]]
```

$$ux2\left(\frac{(x1-x3)\ y}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}+\frac{x\ (-y1+y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}+\frac{x3\ y1-x1\ y3}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}\right)+ux1\left(\frac{(-x2+x3)\ y}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}+\frac{x\ (y2-y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}+\frac{-x3\ y2+x2\ y3}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}\right)+ux3\left(\frac{(-x2+x3)\ y}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}+\frac{x\ (y2-y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}+\frac{-x3\ y2+x2\ y3}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}\right)$$

- Valor del Desplazamiento en y

```
u[x, y][[2, 1]]
```

$$uy2\left(\frac{(x1-x3)\,y}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{x\,(-y1+y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{x3\,y1-x1\,y3}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}\right)+uy1\left(\frac{(-x2+x3)\,y}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{x\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{-x3\,y2+x2\,y3}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}\right)+uy3\left(\frac{(-x2+x3)\,y}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{x\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{-x3\,y2+x2\,y3}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}\right)$$

- Ejemplo:

Coordenadas Nodos Elemento Triangulo Ejemplo

```
CNodos = {x1 → 3, y1 → 4, x2 → 6, y2 → 2, x3 → 6 , y3 → 7 };
```

Valores conocidos Desplazamientos en los Nodos

```
DNodos = {ux1 → 1, uy1 → 3, ux2 → 3, uy2 → 1, ux3 → 2, uy3 → 1};
```

Coordenadas Triangulares de un Punto P del Elemento

```
CTPunto = {ξ1 → 1 / 3, ξ2 → 1 / 3, ξ3 → 1 / 3};
```

Recuerda - Función Transformación Coordenadas Triangulares a Cartesianas

```
TaC[ξ1, ξ2, ξ3] // MatrixForm
```

$$\begin{pmatrix} \xi1+\xi2+\xi3 \\ x1\,\xi1+x2\,\xi2+x3\,\xi3 \\ y1\,\xi1+y2\,\xi2+y3\,\xi3 \end{pmatrix}$$

Coordenadas Cartesianas del Punto P del que se conocen sus Coordenadas Triangulares

```
xp = TaC[ξ1, ξ2, ξ3][[2, 1]];
yp = TaC[ξ1, ξ2, ξ3][[3, 1]];
```

```
xp
```

$x1\,\xi1+x2\,\xi2+x3\,\xi3$

```
xp /. CNodos /. CTPunto
```

5

```
yp /. CNodos /. CTPunto
```

$$\frac{13}{3}$$

Funciones que proporcionan los DESPLAZAMIENTOS en cualquier punto conocidos los nodales

```
u[x, y] /. CNodos /. CTPunto /. DNodos // MatrixForm
```

$$\begin{pmatrix} 2+2\left(2-\frac{x}{3}\right)-\frac{x}{3}+3\left(\frac{1}{5}+\frac{x}{5}-\frac{y}{5}\right) \\ \frac{11}{5}+3\left(2-\frac{x}{3}\right)-\frac{2x}{15}-\frac{y}{5} \end{pmatrix}$$

```
u[xp, yp] /. CNodos /. CTPunto /. DNodos // MatrixForm
```

$$\begin{pmatrix} 2 \\ \frac{5}{3} \end{pmatrix}$$

▪ Obtención de las Deformaciones - 1a Forma: Desplazamientos Puntuales

▫ Relaciones Básicas - Problema Tensión Plana

*** Las deformaciones en el punto se calculan mediante esta relación vectorial en derivadas parciales

```
TensiónPlanaM = Import["014.jpg"];
Show[TensiónPlanaM, ImageSize → 750]
```

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix},$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}, \qquad (14.5)$$

$$\begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

```
TensiónPlanaC = Import["015.jpg"];
Show[TensiónPlanaC, ImageSize → 750]
```

$$\boxed{\mathbf{e} = \mathbf{D}\mathbf{u}, \qquad \sigma = \mathbf{E}\mathbf{e}, \qquad \mathbf{D}^T\sigma + \mathbf{b} = \mathbf{0},} \qquad (14.6)$$

```
Show[Import["0016.jpg"], ImageSize → 1000]
```

§14.3. Linear Elasticity Equations

We shall develop plane stress finite elements in the framework of classical linear elasticity. The necessary governing equations are presented below. They are graphically represented in the Strong Form Tonti diagram of Figure 14.4.

```
Show[Import["0017.jpg"], ImageSize → 1000]
```

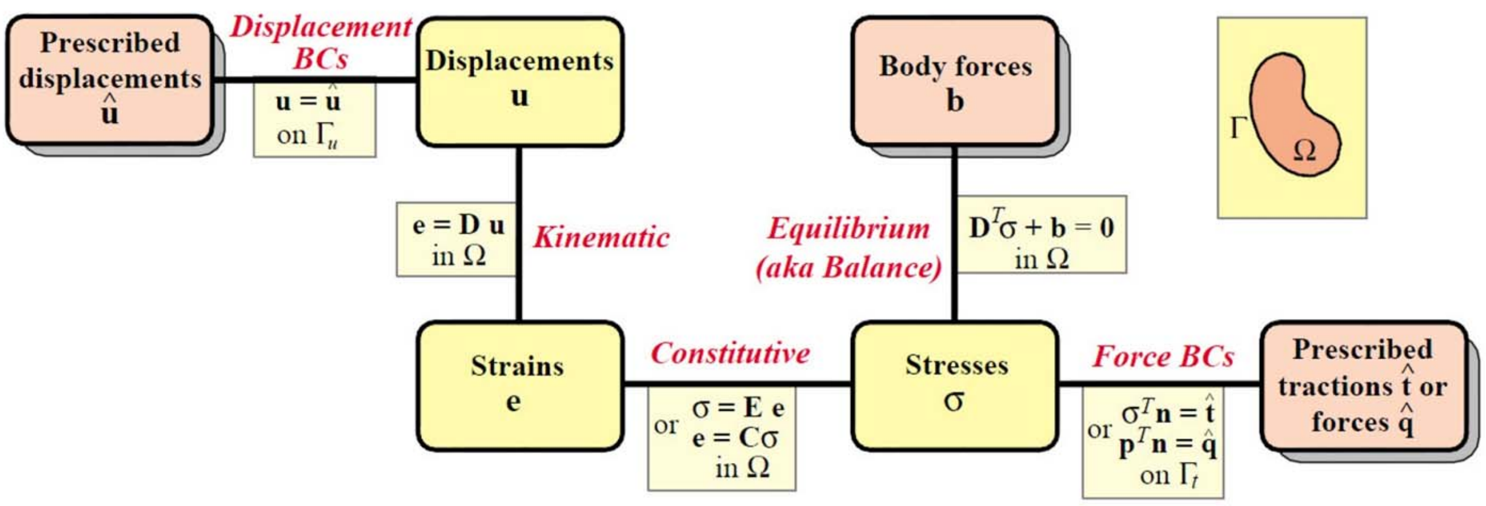


FIGURE 14.4. The Strong Form of the plane stress equations of linear elastostatics displayed as a Tonti diagram. Yellow boxes identify prescribed fields whereas orange boxes denote unknown fields. The distinction between Strong and Weak Forms is explained in §14.3.3.

In[113]:=
```
Show[Import["0018.jpg"], ImageSize → 1000]
```

Out[113]=

§14.3.1. Governing Equations

The three internal fields: displacements, strains and stresses 14.1–14.3 are connected by three field equations: kinematic, constitutive and internal-equilibrium equations. If initial strain effects are ignored, these equations read

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix},$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}, \tag{14.5}$$

$$\begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The compact matrix version of 14.5 is

$$\boxed{\mathbf{e} = \mathbf{D}\,\mathbf{u}, \qquad \boldsymbol{\sigma} = \mathbf{E}\,\mathbf{e}, \qquad \mathbf{D}^T\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0},} \tag{14.6}$$

Here $\mathbf{E} = \mathbf{E}^T$ is the 3×3 stress-strain matrix of plane stress elastic moduli, $\mathbf{D}$ is the 3×2 symmetric-gradient operator and its transpose the 2×3 tensor-divergence operator.[2]

*** Matriz D - Dd

```
?D
```

D[f, x] gives the partial derivative $\partial f / \partial x$.

D[f, {x, n}] gives the multiple derivative $\partial^n f / \partial x^n$.

D[f, x, y, ...] differentiates f successively with respect to x, y, ...

D[f, {{x_1, x_2, ...}}] for a scalar f gives the vector derivative $(\partial f / \partial x_1, \partial f / \partial x_2, \ldots)$.

D[f, {$array$}] gives a tensor derivative. >>

```
D[u[x, y][[1, 1]], x]
```

$$\frac{ux1\,(y2 - y3)}{-x2\,y1 + x3\,y1 + x1\,y2 - x3\,y2 - x1\,y3 + x2\,y3} + \frac{ux3\,(y2 - y3)}{-x2\,y1 + x3\,y1 + x1\,y2 - x3\,y2 - x1\,y3 + x2\,y3} + \frac{ux2\,(-y1 + y3)}{-x2\,y1 + x3\,y1 + x1\,y2 - x3\,y2 - x1\,y3 + x2\,y3}$$

```
∂ₓu[x, y][[1, 1]]
```

$$\frac{ux1\,(y2 - y3)}{-x2\,y1 + x3\,y1 + x1\,y2 - x3\,y2 - x1\,y3 + x2\,y3} + \frac{ux3\,(y2 - y3)}{-x2\,y1 + x3\,y1 + x1\,y2 - x3\,y2 - x1\,y3 + x2\,y3} + \frac{ux2\,(-y1 + y3)}{-x2\,y1 + x3\,y1 + x1\,y2 - x3\,y2 - x1\,y3 + x2\,y3}$$

$$Dd = \begin{pmatrix} ddx & 0 \\ 0 & ddy \\ ddy & ddx \end{pmatrix};$$

$$Dd.\begin{pmatrix} ux \\ uy \end{pmatrix} \text{ // MatrixForm}$$

$$\begin{pmatrix} ddx\,ux \\ ddy\,uy \\ ddy\,ux + ddx\,uy \end{pmatrix}$$

▫ Vector Deformaciones en un Punto P

$$e1[x\_, y\_] = \begin{pmatrix} \partial_x u[x, y][[1, 1]] \\ \partial_y u[x, y][[2, 1]] \\ \partial_y u[x, y][[1, 1]] + \partial_x u[x, y][[2, 1]] \end{pmatrix};$$

```
e1[x, y] // MatrixForm
```

$$\left(\begin{array}{l} \frac{ux1\ (y2-y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{ux3\ (y2-y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{ux2\ (-y1+y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1 \ldots} \\ \frac{uy2\ (x1-x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{uy1\ (-x2+x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{uy3\ (-x2+x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1 \ldots} \\ \frac{ux2\ (x1-x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{ux1\ (-x2+x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{ux3\ (-x2+x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{uy1\ (y2-y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{\ldots}{-x2\ y1+\ldots} \end{array}\right.$$

```
Dimensions[e1[x, y]]
```

{3, 1}

-:) tres filas por una columna

▫ Valor de la Deformación en x - exx

```
e1[x, y][[1, 1]]
```

$$\frac{ux1\ (y2-y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{ux3\ (y2-y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{ux2\ (-y1+y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}$$

▫ Valor de la Deformación en y - eyy

```
e1[x, y][[2, 1]]
```

$$\frac{uy2\ (x1-x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{uy1\ (-x2+x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{uy3\ (-x2+x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}$$

▫ Valor de la Deformación Tangencial xy - exy

```
e1[x, y][[3, 1]] / 2
```

$$\frac{1}{2}\left(\frac{ux2\ (x1-x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{ux1\ (-x2+x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{ux3\ (-x2+x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{uy1\ (y2-y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{uy3\ (y2-y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3} + \frac{uy2\ (-y1+y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}\right)$$

▫ Ejemplo:

```
e1[xp, yp] /. CNodos /. CTPunto /. DNodos // MatrixForm
```

$$\begin{pmatrix} -\frac{2}{5} \\ -\frac{1}{5} \\ -\frac{26}{15} \end{pmatrix}$$

En otro punto

```
CTPunto2 = {ξ1 → 0, ξ2 → 1 / 2, ξ3 → 1 / 2};
```

```
xp /. CNodos /. CTPunto2
```

6

```
yp /. CNodos /. CTPunto2
```

$\frac{9}{2}$

```
e1[xp, yp] /. CNodos /. CTPunto2 /. DNodos // MatrixForm
```

$$\begin{pmatrix} -\frac{2}{5} \\ -\frac{1}{5} \\ -\frac{26}{15} \end{pmatrix}$$

Obtención de las Deformaciones - 2a Forma: Desplazamientos Nodales

Relaciones Básicas - Problema Tensión Plana

*** Las deformaciones en el punto se calculan mediante esta relación vectorial en derivadas parciales

In[114]:=
```
Show[TensiónPlanaM, ImageSize → 750]
```

Out[114]=

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix},$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}, \qquad (14.5)$$

$$\begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In[115]:=
```
Show[TensiónPlanaC, ImageSize → 750]
```

Out[115]=

$$\mathbf{e} = \mathbf{D}\mathbf{u}, \qquad \boldsymbol{\sigma} = \mathbf{E}\mathbf{e}, \qquad \mathbf{D}^T\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}, \qquad (14.6)$$

In[116]:=
```
Show[InterpolaciónFN, ImageSize → 750]
```

Out[116]=

$$\mathbf{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} N_1^{(e)} & 0 & N_2^{(e)} & 0 & \dots & N_n^{(e)} & 0 \\ 0 & N_1^{(e)} & 0 & N_2^{(e)} & \dots & 0 & N_n^{(e)} \end{bmatrix} \mathbf{u}^{(e)} = \mathbf{N}^{(e)}\mathbf{u}^{(e)}. \qquad (14.17)$$

In[117]:=
```
Deformaciones = Import["016.jpg"];
Show[Deformaciones, ImageSize → 750]
```

Out[118]=

$$\mathbf{e}(x,y) = \begin{bmatrix} \frac{\partial N_1^{(e)}}{\partial x} & 0 & \frac{\partial N_2^{(e)}}{\partial x} & 0 & \dots & \frac{\partial N_n^{(e)}}{\partial x} & 0 \\ 0 & \frac{\partial N_1^{(e)}}{\partial y} & 0 & \frac{\partial N_2^{(e)}}{\partial y} & \dots & 0 & \frac{\partial N_n^{(e)}}{\partial y} \\ \frac{\partial N_1^{(e)}}{\partial y} & \frac{\partial N_1^{(e)}}{\partial x} & \frac{\partial N_2^{(e)}}{\partial y} & \frac{\partial N_2^{(e)}}{\partial x} & \dots & \frac{\partial N_n^{(e)}}{\partial y} & \frac{\partial N_n^{(e)}}{\partial x} \end{bmatrix} \mathbf{u}^{(e)} = \mathbf{B}\,\mathbf{u}^{(e)}. \qquad (14.18)$$

```
Nn = ( N11  0    N22  0    N33  0
       0    N11  0    N22  0    N33 );
```

```
( ddx  0
  0    ddy
  ddy  ddx ).Nn // MatrixForm
```

$$\begin{pmatrix} \text{ddx N11} & 0 & \text{ddx N22} & 0 & \text{ddx N33} & 0 \\ 0 & \text{ddy N11} & 0 & \text{ddy N22} & 0 & \text{ddy N33} \\ \text{ddy N11} & \text{ddx N11} & \text{ddy N22} & \text{ddx N22} & \text{ddy N33} & \text{ddx N33} \end{pmatrix}$$

```
Show[Import["0019.jpg"], ImageSize → 1000]
```

Differentiating the finite element displacement field yields the strain-displacement relations:

$$\mathbf{e}(x,y)=\begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \dots & \frac{\partial N_n^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & \dots & 0 & \frac{\partial N_n^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \dots & \frac{\partial N_n^e}{\partial y} & \frac{\partial N_n^e}{\partial x} \end{bmatrix} \mathbf{u}^e = \mathbf{B}\,\mathbf{u}^e. \qquad (14.18)$$

This $\mathbf{B} = \mathbf{D}\,\mathbf{N}$ is called the *strain-displacement matrix*. It is dimensioned $3 \times 2n$. For example, if the element has 6 nodes, $\mathbf{B}$ is 3×12. The stresses are given in terms of strains and displacements by $\boldsymbol{\sigma} = \mathbf{E}\,\mathbf{e} = \mathbf{E}\mathbf{B}\mathbf{u}^e$, which is assumed to hold at all points of the element.

```
Show[Import["0020.jpg"], ImageSize → 1000]
```

§15.3.2. Strain-Displacement Equations

The strains within the elements are obtained by differentiating the shape functions with respect to x and y. Using 15.14, 15.16 and the general form (14.18) we get

$$\mathbf{e} = \mathbf{D}\,\mathbf{N}\,\mathbf{u}^e = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{B}\,\mathbf{u}^e, \qquad (15.17)$$

in which $\mathbf{D}$ denotes the symbolic strain-to-displacement differentiation operator given in (14.6), and $\mathbf{B}$ is the strain-displacement matrix. Note that the strains are *constant* over the element. This is the origin of the name *constant strain triangle* (CST) given it in many finite element publications.

***** Matriz B - Be

$$Be = \begin{pmatrix} \partial_x N1 & 0 & \partial_x N2 & 0 & \partial_x N3 & 0 \\ 0 & \partial_y N1 & 0 & \partial_y N2 & 0 & \partial_y N3 \\ \partial_y N1 & \partial_x N1 & \partial_y N2 & \partial_x N2 & \partial_y N3 & \partial_x N3 \end{pmatrix};$$

Vector Deformaciones en un Punto P

```
e2[x_, y_] = Be.ue;
```

```
e2[x, y] // MatrixForm
```

$$\begin{pmatrix} \frac{ux1\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{ux3\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{ux2\,(-y1+y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1 \dots} \\ \frac{uy2\,(x1-x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{uy1\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{uy3\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1 \dots} \\ \frac{ux2\,(x1-x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{ux1\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{ux3\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{uy1\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{\dots}{-x2\,y1+ \dots} \end{pmatrix}$$

```
Dimensions[e2[x, y]]
```

{3, 1}

-:) tres filas por una columna

Valor de la Deformación en x - exx

```
e2[x, y][[1, 1]]
```

$$\frac{ux1\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{ux3\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3} + \frac{ux2\,(-y1+y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}$$

▫ Valor de la Deformación en y - eyy

```
e2[x, y][[2, 1]]
```

$$\frac{uy2\ (x1-x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}+\frac{uy1\ (-x2+x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}+\frac{uy3\ (-x2+x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}$$

▫ Valor de la Deformación Tangencial xy - exy

```
e2[x, y][[3, 1]] / 2
```

$$\frac{1}{2}\left(\frac{ux2\ (x1-x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}+\frac{ux1\ (-x2+x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}+\frac{ux3\ (-x2+x3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}+\frac{uy1\ (y2-y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}+\frac{uy3\ (y2-y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}+\frac{uy2\ (-y1+y3)}{-x2\ y1+x3\ y1+x1\ y2-x3\ y2-x1\ y3+x2\ y3}\right)$$

■ Obtención de las Tensiones - Tensión Plana

▫ Relaciones Básicas - Problema Tensión Plana

In[121]:=
```
Tensiones = Import["017.jpg"];
Show[Tensiones, ImageSize → 750]
```

Out[122]=

$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix}=\begin{bmatrix}E_{11} & E_{12} & E_{13}\\ E_{12} & E_{22} & E_{23}\\ E_{13} & E_{23} & E_{33}\end{bmatrix}\begin{bmatrix}e_{xx}\\ e_{yy}\\ 2e_{xy}\end{bmatrix}=\mathbf{E}\mathbf{e},\qquad(15.19)$$

In[123]:=
```
TensionesTP = Import["018.jpg"];
Show[TensionesTP, ImageSize → 750]
```

Out[124]=

[A:25] Suppose that the structural material is isotropic, with elastic modulus E and Poisson's ratio ν. The in-plane stress-strain relations for plane stress ($\sigma_{zz}=\sigma_{xz}=\sigma_{yz}=0$) and plane strain ($e_{zz}=e_{xz}=e_{yz}=0$) as given in any textbook on elasticity, are

$$\text{plane stress:}\quad \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}e_{xx}\\ e_{yy}\\ 2e_{xy}\end{bmatrix},$$

$$\text{plane strain:}\quad \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix}1 & \frac{\nu}{1-\nu} & 0\\ \frac{\nu}{1-\nu} & 1 & 0\\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)}\end{bmatrix}\begin{bmatrix}e_{xx}\\ e_{yy}\\ 2e_{xy}\end{bmatrix}.\qquad(E14.1)$$

In[125]:=
```
Show[Import["0021.jpg"], ImageSize → 1000]
```

Out[125]=

§15.3.3. Stress-Strain Equations

The stress field $\boldsymbol{\sigma}$ is related to the strain field by the elastic constitutive equation in (14.5), which is repeated here for convenience:

$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix}=\begin{bmatrix}E_{11} & E_{12} & E_{13}\\ E_{12} & E_{22} & E_{23}\\ E_{13} & E_{23} & E_{33}\end{bmatrix}\begin{bmatrix}e_{xx}\\ e_{yy}\\ 2e_{xy}\end{bmatrix}=\mathbf{E}\,\mathbf{e},\qquad(15.18)$$

where E_{ij} are plane stress elastic moduli. The constitutive matrix $\mathbf{E}$ will be assumed to be constant over the element. Because the strains are constant, so are the stresses.

▫ ***** Matriz del Material E - Em

$$Ee=\frac{Em}{1-\nu^2}*\begin{pmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{pmatrix};$$

▫ Vector Tensiones en un Punto P

```
σ[x_, y_] = Ee.e[x, y];
```

```
σ[x, y] // MatrixForm
```

$$\begin{pmatrix} \frac{Em\left(\frac{ux1\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{ux3\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{ux2\,(-y1+y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}\right)}{1-\nu^2}+\frac{Em\left(\frac{uy2\,(x1-x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy1\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy3\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2}\right.}{1-\nu^2} \\ \frac{Em\left(\frac{uy2\,(x1-x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy1\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy3\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}\right)}{1-\nu^2}+\frac{Em\left(\frac{ux1\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{ux3\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{ux2\,(-y1+y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2}\right.}{1-\nu^2} \\ \frac{Em\left(\frac{ux2\,(x1-x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{ux1\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{ux3\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy1\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy3\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy2\,(-y1+y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2}\right.}{2\left(1-\nu^2\right)} \end{pmatrix}$$

```
Dimensions[e2[x, y]]
```

{3, 1}

-:) tres filas por una columna

▫ Valor de la Tensión en x - σxx

```
σ[x, y][[1, 1]]
```

$$\frac{1}{1-\nu^2}Em\left(\frac{ux1\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{ux3\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{ux2\,(-y1+y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}\right)+\frac{1}{1-\nu^2}Em\left(\frac{uy2\,(x1-x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy1\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy3\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}\right)\nu$$

▫ Valor de la Tensión en y - σyy

```
σ[x, y][[2, 1]]
```

$$\frac{1}{1-\nu^2}Em\left(\frac{uy2\,(x1-x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy1\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy3\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}\right)+\frac{1}{1-\nu^2}Em\left(\frac{ux1\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{ux3\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{ux2\,(-y1+y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}\right)\nu$$

▫ Valor de la Tensión Tangencial xy - σxy

```
σ[x, y][[3, 1]] / 2
```

$$\frac{1}{4\left(1-\nu^2\right)}Em\left(\frac{ux2\,(x1-x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{ux1\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{ux3\,(-x2+x3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy1\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy3\,(y2-y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}+\frac{uy2\,(-y1+y3)}{-x2\,y1+x3\,y1+x1\,y2-x3\,y2-x1\,y3+x2\,y3}\right)(1-\nu)$$

▫ Ejemplo:

```
Material = {Em → 1000, ν → 1 / 3};
```

```
σ[xp, yp] /. CNodos /. CTPunto /. DNodos /. Material // MatrixForm
```

$$\begin{pmatrix} -525 \\ -375 \\ -650 \end{pmatrix}$$

▪ Matriz de Rigidez

In[126]:=
```
MatrizRigidez = Import["019.jpg"];
Show[MatrizRigidez, ImageSize → 750]
```

Out[127]=

$$\mathbf{K}^{(e)} = \int_{\Omega^{(e)}} h\, \mathbf{B}^T \mathbf{E} \mathbf{B}\, d\Omega^{(e)}, \qquad (15.20)$$

where $\Omega^{(e)}$ is the triangle domain, and h is the plate thickness that appears in the plane stress problem. Since **B** and **E** are constant, they can be taken out of the integral:

In[128]:=
```
MatrizRigidezh = Import["020.jpg"];
Show[MatrizRigidezh, ImageSize → 750]
```

Out[129]=

$$\mathbf{K}^{(e)} = \mathbf{B}^T \mathbf{E} \mathbf{B} \int_{\Omega^{(e)}} h\, d\Omega^{(e)}$$

Si el espesor del elemento h es constante la MATRIZ DE RIGIDEZ será:

```
Ke[x_, y_] = Transpose[Be].Ee.Be * AreaT[x1, y1, x2, y2, x3, y3];
```

```
Ke[x_, y_] = FullSimplify[Ke[x, y]];
```

```
Ke[x, y] // MatrixForm
```

$$\begin{pmatrix}
\frac{Em\,(2\,(y2-y3)^2-(x2-x3)^2\,(-1+\nu))}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)+x1\,(-y2+y3))\,(-1+\nu^2)} & -\frac{Em\,(x2-x3)\,(y2-y3)}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)+x1\,(-y2+y3))\,(-1+\nu)} & \frac{Em\,(2\,(y1-y3)\,(-y2+y3)+(x1\ldots}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)-\ldots} \\
-\frac{Em\,(x2-x3)\,(y2-y3)}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)+x1\,(-y2+y3))\,(-1+\nu)} & \frac{Em\,(2\,(x2-x3)^2-(y2-y3)^2\,(-1+\nu))}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)+x1\,(-y2+y3))\,(-1+\nu^2)} & \frac{1}{4}\,Em\,\left(\frac{(x2-x3)\,(y\ldots}{(x3\,(-y1+y2)+x2\,(y1-y3)+\ldots}\right. \\
\frac{Em\,(2\,(y1-y3)\,(-y2+y3)+(x1-x3)\,(x2-x3)\,(-1+\nu))}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)+x1\,(-y2+y3))\,(-1+\nu^2)} & \frac{1}{4}\,Em\,\left(\frac{(x2-x3)\,(y1-y3)}{(x3\,(-y1+y2)+x2\,(y1-y3)+x1\,(-y2+y3))\,(-1+\nu)}+\frac{1}{1+\nu}\right) & \frac{Em\,(2\,(y1-y3)^2-(x1\ldots}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)-\ldots} \\
\frac{1}{4}\,Em\,\left(\frac{(x1-x3)\,(y2-y3)}{(x3\,(-y1+y2)+x2\,(y1-y3)+x1\,(-y2+y3))\,(-1+\nu)}-\frac{1}{1+\nu}\right) & \frac{Em\,(2\,(x1-x3)\,(-x2+x3)+(y1-y3)\,(y2-y3)\,(-1+\nu))}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)+x1\,(-y2+y3))\,(-1+\nu^2)} & \frac{Em\,(-x1+x3)\ldots}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)\ldots} \\
\frac{Em\,(2\,(y2-y3)^2-(x2-x3)^2\,(-1+\nu))}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)+x1\,(-y2+y3))\,(-1+\nu^2)} & -\frac{Em\,(x2-x3)\,(y2-y3)}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)+x1\,(-y2+y3))\,(-1+\nu)} & \frac{Em\,(2\,(y1-y3)\,(-y2+y3)+(x1\ldots}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)-\ldots} \\
-\frac{Em\,(x2-x3)\,(y2-y3)}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)+x1\,(-y2+y3))\,(-1+\nu)} & \frac{Em\,(2\,(x2-x3)^2-(y2-y3)^2\,(-1+\nu))}{4\,(x3\,(-y1+y2)+x2\,(y1-y3)+x1\,(-y2+y3))\,(-1+\nu^2)} & \frac{1}{4}\,Em\,\left(\frac{(x2-x3)\,(y\ldots}{(x3\,(-y1+y2)+x2\,(y1-y3)+\ldots}\right.
\end{pmatrix}$$

▫ Ejemplo:

```
Ke[xp, yp] /. CNodos /. Material /. CTPunto // MatrixForm
```

$$\begin{pmatrix}
\frac{1875}{2} & 0 & -\frac{1125}{2} & \frac{375}{2} & \frac{1875}{2} & 0 \\
0 & \frac{625}{2} & \frac{375}{2} & -\frac{375}{2} & 0 & \frac{625}{2} \\
-\frac{1125}{2} & \frac{375}{2} & 450 & -225 & -\frac{1125}{2} & \frac{375}{2} \\
\frac{375}{2} & -\frac{375}{2} & -225 & 450 & \frac{375}{2} & -\frac{375}{2} \\
\frac{1875}{2} & 0 & -\frac{1125}{2} & \frac{375}{2} & \frac{1875}{2} & 0 \\
0 & \frac{625}{2} & \frac{375}{2} & -\frac{375}{2} & 0 & \frac{625}{2}
\end{pmatrix}$$

```
Eigenvalues[%]
```

$$\left\{75\left(17+3\sqrt{21}\right),\ 850,\ 75\left(17-3\sqrt{21}\right),\ 0,\ 0,\ 0\right\}$$