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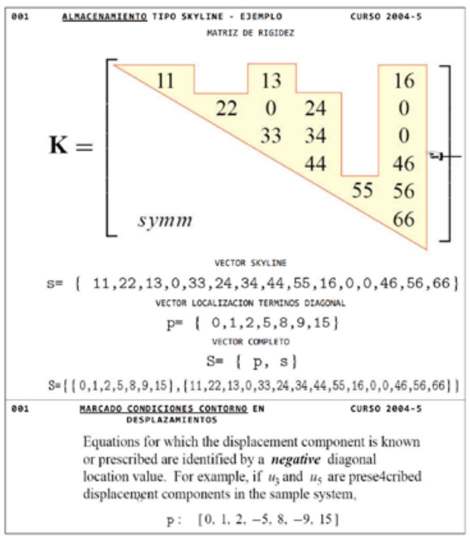
Estas ACTIVIDADES DE CLASE deberá realizarse descargando los documentos *NB* disponibles en las páginas web, completandolos adecuadamente, denominandolos de la forma especificada y subiendolos a tu cuenta de entrega personal. En este documento *PDF* habrá que contestar a las *PREGUNTAS* que planteo a lo largo de la grabación en video correspondiente a la clase.

Para familiarizarnos con la ***Implementación del Elemento Cuadrilátero***, su definición, su terminología y su planteamiento; durante las explicaciones en clase habrá que completar este documento PDF.

Estas son imágenes de algunos de los ejercicios considerados en las ACTIVIDADES de esta CLASE:

07-C7-Mathematica-C


001 ALMACENAMIENTO TIPO SKYLINE - EJEMPLO CURSO 2004-5

MATRIZ DE RIGIDEZ

$$K = \begin{bmatrix} 11 & & 13 & & & 16 \\ & 22 & 0 & 24 & & 0 \\ & & 33 & 34 & & 0 \\ & & & 44 & & 46 \\ & & & & 55 & 56 \\ symm & & & & & 66 \end{bmatrix}$$

VECTOR SKYLINE

s= { 11,22,13,0,33,24,34,44,55,16,0,0,46,56,66}

VECTOR LOCALIZACION TERMINOS DIAGONAL

p= { 0,1,2,5,8,9,15}

VECTOR COMPLETO

S= { p, s}

S={{0,1,2,5,8,9,15},{11,22,13,0,33,24,34,44,55,16,0,0,46,56,66}}

001 MARCADO CONDICIONES CONTORNO EN DESPLAZAMIENTOS CURSO 2004-5

Equations for which the displacement component is known or prescribed are identified by a ***negative*** diagonal location value. For example, if u_3 and u_5 are prese4cribed displacement components in the sample system,

p: [0, 1, 2, −5, 8, −9, 15]

PREGUNTAS Y TUS CONTESTACIONES:

DOCUMENTO PDF A COMPLETAR:

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CONVERGENCIA

001 SIGNIFICADO DEL TERMINO "CONVERGENCIA" CURSO 2004-5

Convergence: discrete (FEM) solution approaches the analytical (math model) solution in some sense

Convergence = +

(Lax-Wendroff)

001 SIGNIFICADO DE LOS TERMINOS "CONSISTENCIA" Y "ESTABILIDAD" CURSO 2004-5

- **Consistency**

  individual elements

  element patches

- **Stability**

  individual elements

  individual elements

Completeness. The elements must have enough *approximation power* to capture the analytical solution in the limit of a mesh refinement process.

Compatibility. The shape functions must provide *displacement continuity* between elements.

Completeness and compatibility are two aspects of the so-called **consistency** condition between the discrete and mathematical models.

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001 SIGNIFICADO DEL "INDICE VARIACIONAL" CURSO 2004-5

The FEM is based on the direct discretization of an energy functional $\Pi[u]$, where u (displacements for the elements considered in this book) is the primary variable, or (equivalently) the function to be varied. Let m be the highest spatial derivative order of u that appears in Π. This m is called the *variational index*.

Total Potential Energy of Plate in Plane Stress

$$\Pi = U - W$$

$$U = \tfrac{1}{2}\int_{\Omega} h\, \boldsymbol{\sigma}^T \mathbf{e}\, d\Omega = \tfrac{1}{2}\int_{\Omega} h\, \mathbf{e}^T \mathbf{E}\mathbf{e}\, d\Omega$$

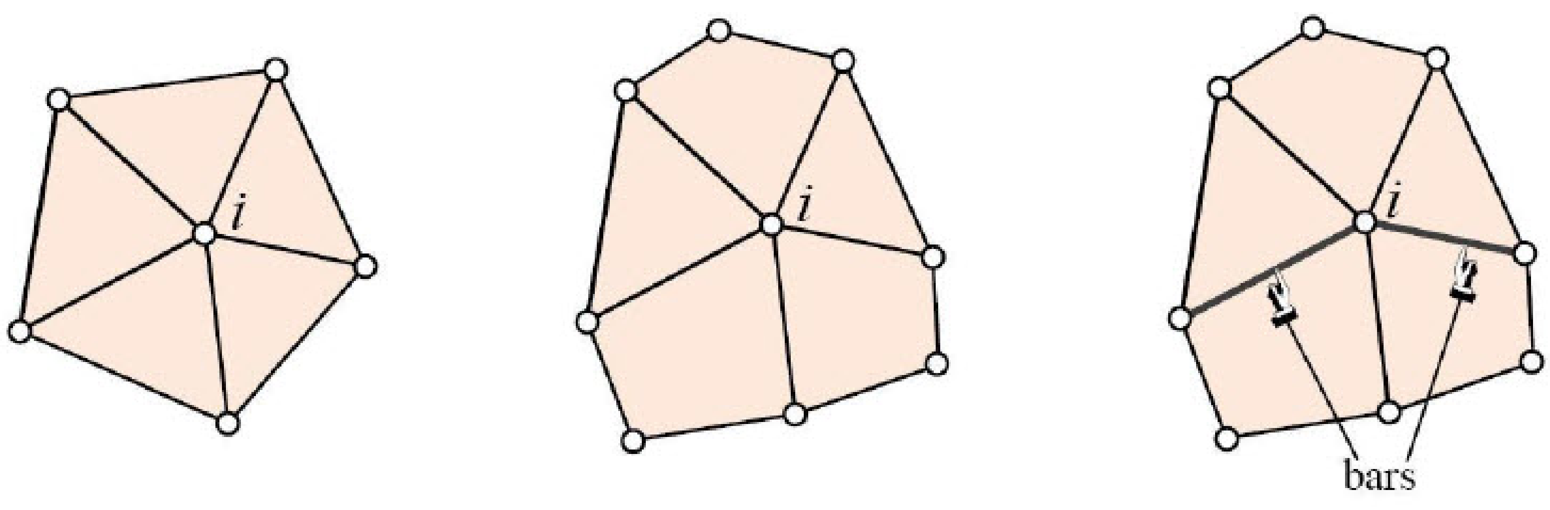
$$W = \int_{\Omega} h\, \mathbf{u}^T \mathbf{b}\, d\Omega + \int_{\Gamma_t} h\, \mathbf{u}^T \hat{\mathbf{t}}\, d\Gamma$$

$$\mathbf{e}(x,y) = \begin{bmatrix} \frac{\partial N_1^{(e)}}{\partial x} & 0 & \frac{\partial N_2^{(e)}}{\partial x} & 0 & \dots & \frac{\partial N_n^{(e)}}{\partial x} & 0 \\ 0 & \frac{\partial N_1^{(e)}}{\partial y} & 0 & \frac{\partial N_2^{(e)}}{\partial y} & \dots & 0 & \frac{\partial N_n^{(e)}}{\partial y} \\ \frac{\partial N_1^{(e)}}{\partial y} & \frac{\partial N_1^{(e)}}{\partial x} & \frac{\partial N_2^{(e)}}{\partial y} & \frac{\partial N_2^{(e)}}{\partial x} & \dots & \frac{\partial N_n^{(e)}}{\partial y} & \frac{\partial N_n^{(e)}}{\partial x} \end{bmatrix} \mathbf{u}^{(e)} = \mathbf{B}\mathbf{u}^{(e)}$$

Plane Stress: $m =$ in **Two Dimensions**

001 DEFINICION DE UN "PATCH" DE ELEMENTOS CURSO 2004-5

A ***patch*** is the set of all elements attached to a given node:



A finite element ***patch trial function*** is the union of shape functions activated by setting a degree of freedom at that node to unity, while all other freedoms are zero. A patch trial function "propagates" only over the patch, and is zero beyond it.

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001 1º REQUERIMIENTO FUNDAMENTAL A CUMPLIR POR LAS FUNCIONES DE FORMA, EXPRESADOS EN TERMINOS DEL "INDICE VARIACIONAL" **CURSO 2004-5**

The ***element shape functions*** must represent exactly all polynomial terms of order $\leq m$ in the Cartesian coordinates. A set of shape functions that satisfies this condition is call m-complete

Note that this requirement applies *at the element level* and involves *all* shape functions of the element.

Plane Stress: $m = 1$ in Two Dimensions

The ***element shape functions*** must represent exactly all polynomial terms of order ≤ 1 in the Cartesian coordinates. That means any *linear polynomial* in x, y with a *constant* as special case

Suppose a displacement-based element is for a plane stress problem, in which $m = 1$. Then 1-completeness requires that the linear displacement field

$$u_x = \alpha_0 + \alpha_1 x + \alpha_2 y, \qquad u_y = \alpha_0 + \alpha_1 x + \alpha_2 y \tag{19.3}$$

be exactly represented for any value of the α coefficients. This is done by evaluating (19.3) at the nodes to form a displacement vector $\mathbf{u}^{(e)}$ and then checking that $\mathbf{u} = \mathbf{N}^{(e)}\mathbf{u}^{(e)}$ recovers exactly (19.3).

The analysis shows

that completeness is satisfied if the

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001 **2º REQUERIMIENTO FUNDAMENTAL A CUMPLIR POR LAS FUNCIONES DE FORMA, EXPRESADOS EN TERMINOS DEL "INDICE VARIACIONAL"** **CURSO 2004-5**

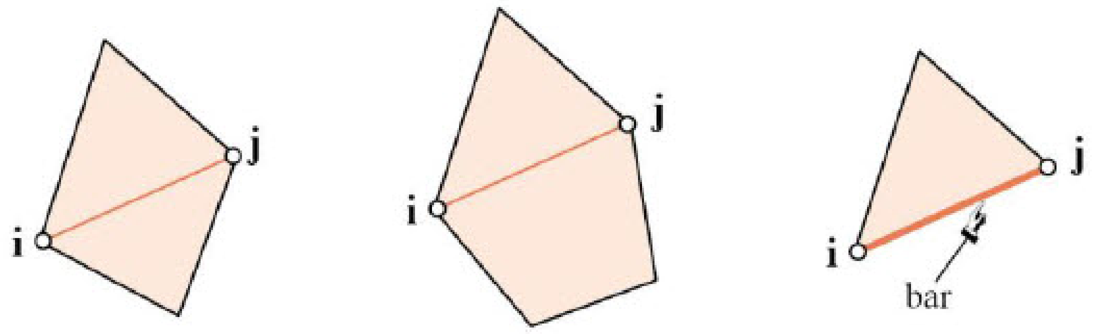
The ***patch trial functions*** must be $C^{(m-1)}$ continuous between elements, and C^{m} piecewise differentiable inside each element

Plane Stress: $m = 1$ in Two Dimensions

The ***patch trial functions*** must be C^{0} continuous between elements, and C^{1} piecewise differentiable inside each element

Interelement Continuity is the Toughest to Meet

Simplification: for *matching meshes* (defined in Notes) it is sufficient to check a *pair of adjacent elements*:



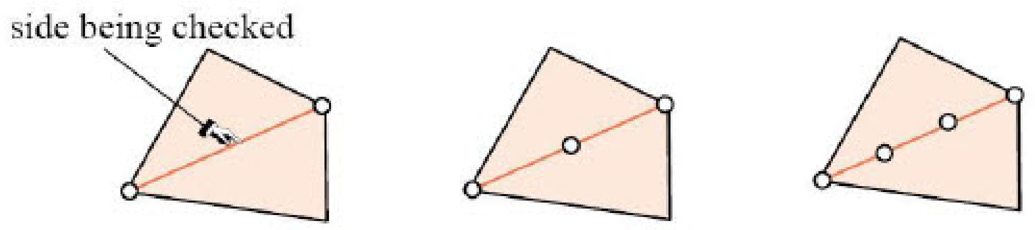
Two 3-node linear triangles

One 3-node linear triangle and one 4-node bilinear quad

One 3-node linear triangle and one 2-node bar

Side Continuity Check for Plane Stress Elements with Polynomial Shape Functions in Natural Coordinates

Let k be the number of nodes on a side:



$k = 2$ $k = 3$ $k = 4$

If *more, continuity is violated*

If *less, nodal configuration is wrong* (too many nodes)

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001 1º REQUERIMIENTO A CUMPLIR POR LA MATRIZ DE RIGIDEZ DEL ELEMENTO PARA ASEGURAR LA "ESTABILIDAD" "SUFICIENCIA DE RANGO" **CURSO 2004-5**

The element stiffness matrix must not possess any zero-energy kinematic mode other than rigid body modes.

This can be mathematically expressed as follows. Let be the number of element degrees of freedom, and be the number of independent rigid body modes. Let r denote the rank of $\mathbf{K}^{(e)}$. The element is called *rank sufficient* if $r = n_F - n_R$ and *rank deficient* if $r < n_F - n_R$. In the latter case,

$$d = (n_F - n_R) - r \tag{19.5}$$

is called the rank deficiency.

If an isoparametric element is numerically integrated, let be the number of Gauss points, while denotes the order of the stress-strain matrix $\mathbf{E}$. Two additional assumptions are made:

(i) The element shape functions satisfy completeness in the sense that the rigid body modes are exactly captured by them.

(ii) Matrix $\mathbf{E}$ is of full rank.

Then each Gauss point adds n_E to the rank of $\mathbf{K}^{(e)}$, up to a maximum of $n_F - n_R$. Hence the rank of $\mathbf{K}^{(e)}$ will be

$$r = \min(n_F - n_R, n_E\, n_G) \tag{19.6}$$

To attain rank sufficiency, $n_E n_G$ must equal or exceed $n_F - n_R$:

(19.7)

from which the appropriate Gauss integration rule can be selected.

In the plane stress problem, $n_E = 3$ because $\mathbf{E}$ is a 3×3 matrix of elastic moduli; see Chapter 14. Also $n_R = 3$. Consequently $r = \min(n_F - 3, 3n_G)$ and $3n_G \geq n_F - 3$.

EXAMPLE 19.5

Consider a plane stress 6-node quadratic triangle. Then $n_F = 2 \times 6 = 12$. To attain the proper rank of $12 - n_R = 12 - 3 = 9$, $n_G \geq 3$. A 3-point Gauss rule, such as the midpoint rule defined in §24.2, makes the element rank sufficient.

EXAMPLE 19.6

Consider a plane stress 9-node biquadratic quadrilateral. Then $n_F = 2 \times 9 = 18$. To attain the proper rank of $18 - n_R = 18 - 3 = 15$, $n_G \geq 5$. The 2×2 product Gauss rule is insufficient because $n_G = 4$. Hence a 3×3 rule, which yields $n_G = 9$, is required to attain rank sufficiency.

Table 19.1 collects rank-sufficient Gauss integration rules for some widely used plane stress elements with n nodes and $n_F = 2n$ freedoms.

Element	n	n_F	$n_F - 3$	Min n_G	Recommended rule
3-node triangle	3	6	3		centroid*
6-node triangle	6	12	9		3-midpoint rule*
10-node triangle	10	20	17		7-point rule*
4-node quadrilateral	4	8	5		2 x 2
8-node quadrilateral	8	16	13		3 x 3
9-node quadrilateral	9	18	15		3 x 3
16-node quadrilateral	16	32	29		4 x 4

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001 **2º REQUERIMIENTO A CUMPLIR POR LA GEOMETRIA DEL ELEMENTO PARA ASEGURAR LA "ESTABILIDAD" "JACOBIANO POSITIVO"** **CURSO 2004-5**

The geometry of the element must be such that the determinant $J = \det \mathbf{J}$ of the Jacobian matrix defined[4] in §17.2, is positive everywhere. As illustrated in Equation (17.20), J characterizes the local metric of the element natural coordinates.

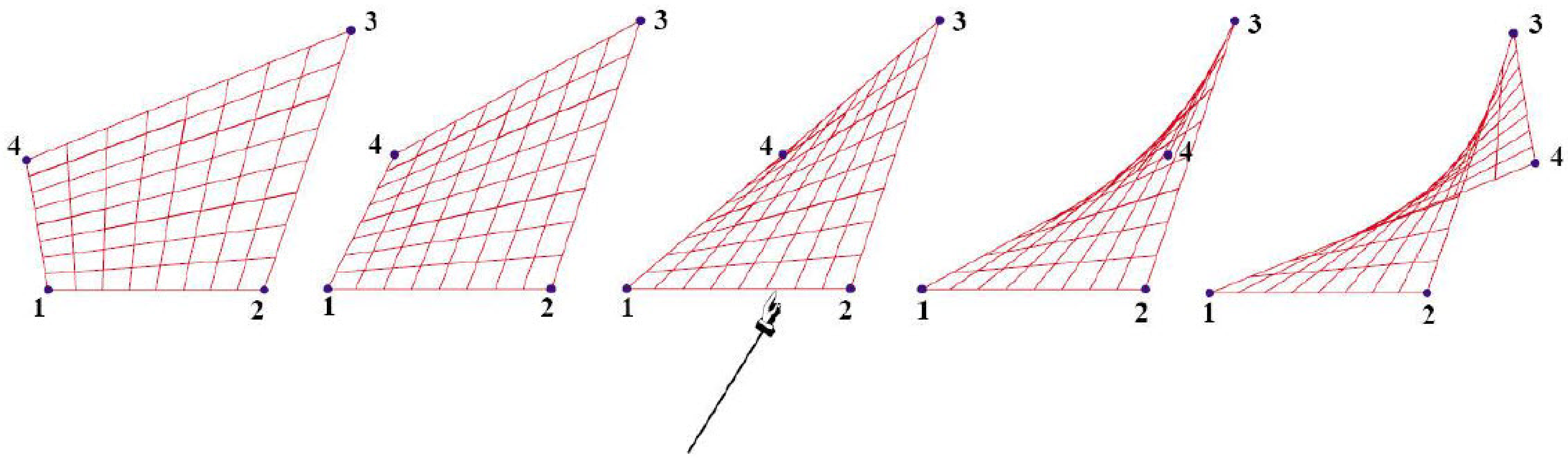
001 **TRIANGULO DE 3 NODOS** **CURSO 2004-5**

For a three-node triangle J is constant and in fact equal to $2A$. The requirement $J > 0$ is equivalent [illegible]

This is called a *convexity condition*. It is easily checked by a finite element program.

001 **CUADRILATERO DE 4 NODOS** **CURSO 2004-5**

But for 2D elements with more than 3 nodes distortions may render *portions* of the element metric negative. This is illustrated in Figure 19.2 for a 4-node quadrilateral in which node 4 is gradually moved to the right. The quadrilateral morphs from a convex figure into a nonconvex one. The center figure is a triangle; note that the metric near node 4 is badly distorted (in fact $J = 0$ there) rendering the element unacceptable. This contradicts the (erroneous) advise of some FE books, which state that quadrilaterals can be reduced to triangles as special cases, thereby rendering triangular elements unnecessary.



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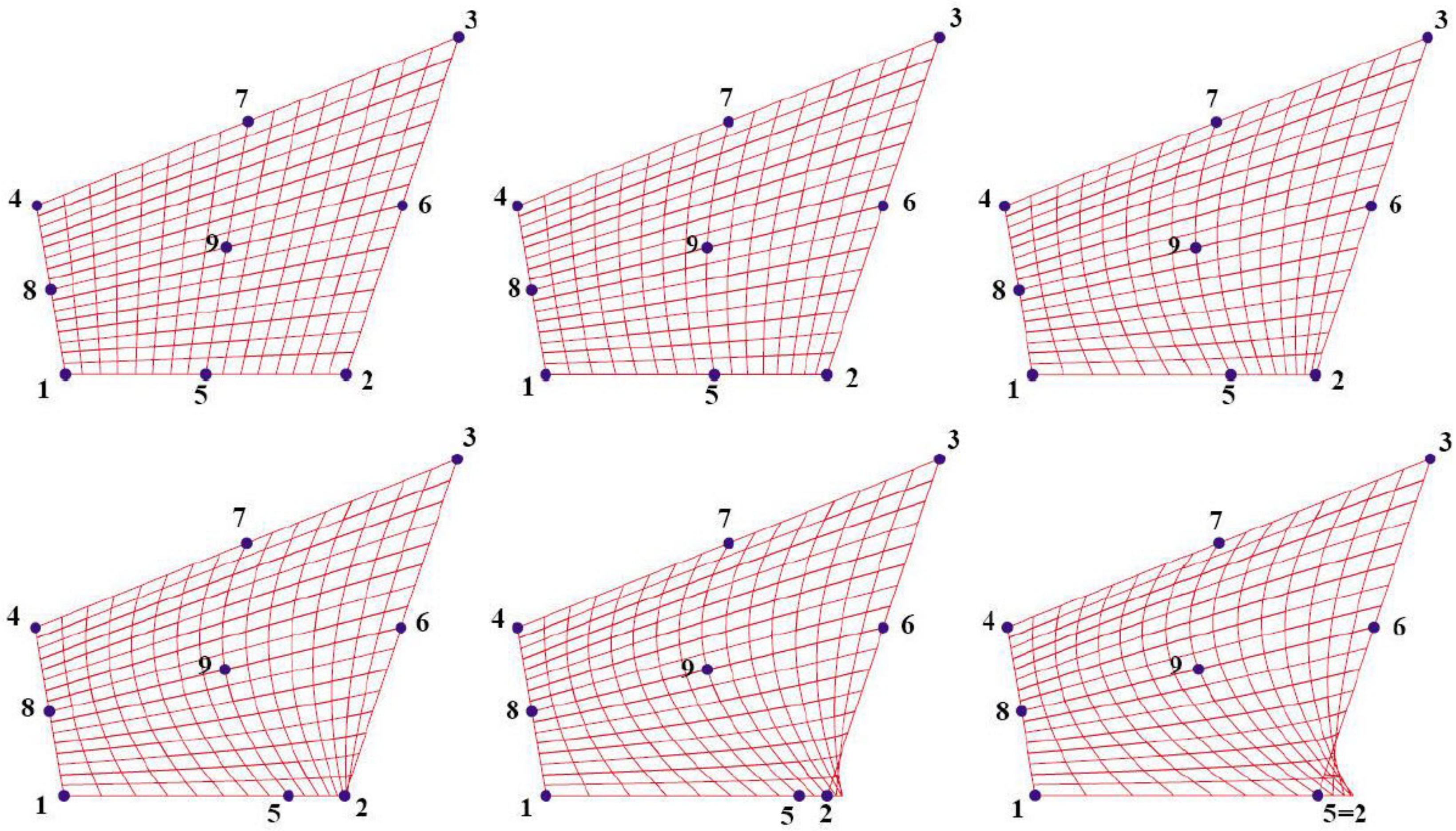
001 CUADRILATERO DE 9 NODOS CURSO 2004-5

For higher order elements proper location of corner nodes is not enough.

The effect of midpoint motions in quadratic elements is illustrated in Figures 19.3 and 19.4.

Figure 19.3 depicts the effect of moving midside node 5 tangentially in a 9-node quadrilateral element while keeping all other 8 nodes fixed. When the location of 5 reaches the quarter-point of side 1-2, the metric at corner 2 becomes singular in the sense that $J = 0$ there. Although this is disastrous in ordinary FE work, it has applications in the construction of special "crack" elements for linear fracture mechanics.



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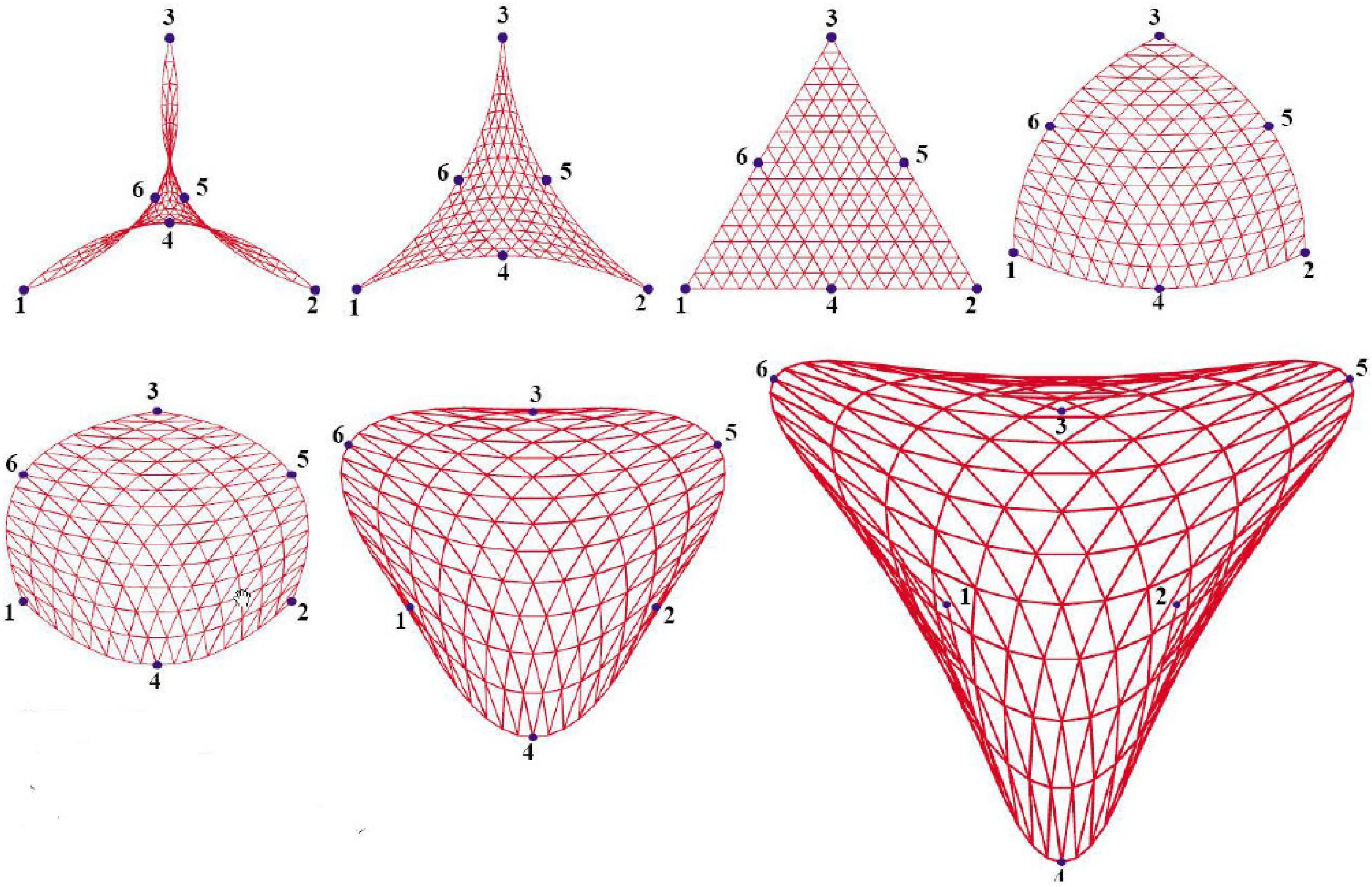
001 **TRIANGULO DE SEIS NODOS** **CURSO 2004-5**

…, as illustrated in Figure 19.4. This depicts a 6-node equilateral triangle in which midside nodes 4, 5 and 6 are moved inwards and outwards along the normals to the midpoint location. As shown in the lower left picture, the element may be even morphed into a "parabolic circle" without the metric breaking down.



001 **EJERCICIO 1** **CURSO 2004-5**

EXERCISE 19.1

[D:15] Draw a picture of a 2D non-matching mesh in which element nodes on two sides of a boundary do not share the same locations. Discuss why enforcing compatibility becomes difficult.

001 **EJERCICIO 2** **CURSO 2004-5**

EXERCISE 19.4

[A:20] Consider three dimensional solid "brick" elements with n nodes and 3 degrees of freedom per node so $n_F = 3n$. The correct number of rigid body modes is 6. Each Gauss integration point adds 6 to the rank; that is, $N_E = 6$. By applying (19.7), find the minimal rank-preserving Gauss integration rules with p points in each direction (that is, $1\times1\times1$, $2\times2\times2$, etc) if the number of node points is n = 8, 20, 27, or 64.

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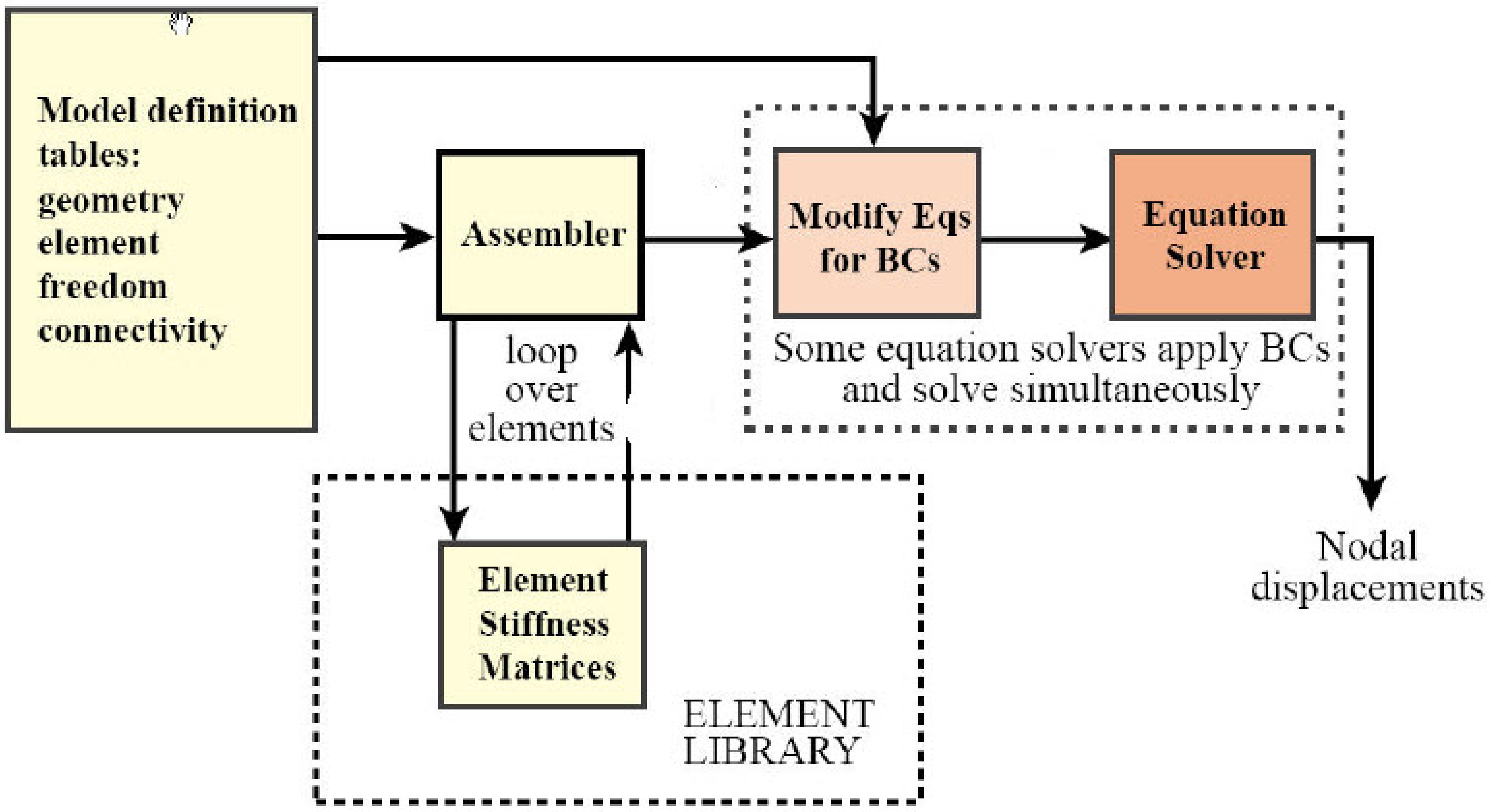
SOLUCION ECUACIONES

001 **PROBLEMA MATEMATICA A RESOLVER** **CURSO 2004-5**

001 **VISUALIZACION DEL PROCESO GENERAL DE ANALISIS POR EF** **CURSO 2004-5**



001 **RECURSOS COMPUTACIONALES NECESARIOS SOLUCION ECUACIONES - MATRIZ COMPLETA** **CURSO 2004-5**

Storage and Solution Times for a Fully Stored Stiffness Matrix

Matrix order N	Storage (double prec)	Factor op. units	Factor time workstation/PC	Factor time supercomputer
10^4		$10^{12}/6$	3 hrs	2 min
10^5	80 GB	$10^{15}/6$	4 mos	30 hrs
10^6	8 TB	$10^{18}/6$	300 yrs	3 yrs

time numbers last adjusted in 1998
to get current times divide by 10-20

As regards memory needs, a full square matrix stored without taking advantage of symmetry, requires storage for N^2 entries. If each entry is an 8-byte, double precision floating-point number, the required storage is $8N^2$ bytes. Thus, a matrix of order $N = 10^4$ would require 8×10^8 bytes or 800 MegaBytes (MB) for storage.

For large N the solution of (26.1) is dominated by the factorization of **K**, an operation discussed in §26.2. This operation requires approximately $N^3/6$ floating point operation units. [A floating-point operation unit is conventionally defined as a (multiply,add) pair plus associated indexing and data movement operations.] Now a fast workstation can typically do 10^7 of these operations per second, whereas a supercomputer may be able to sustain 10^9 or more.

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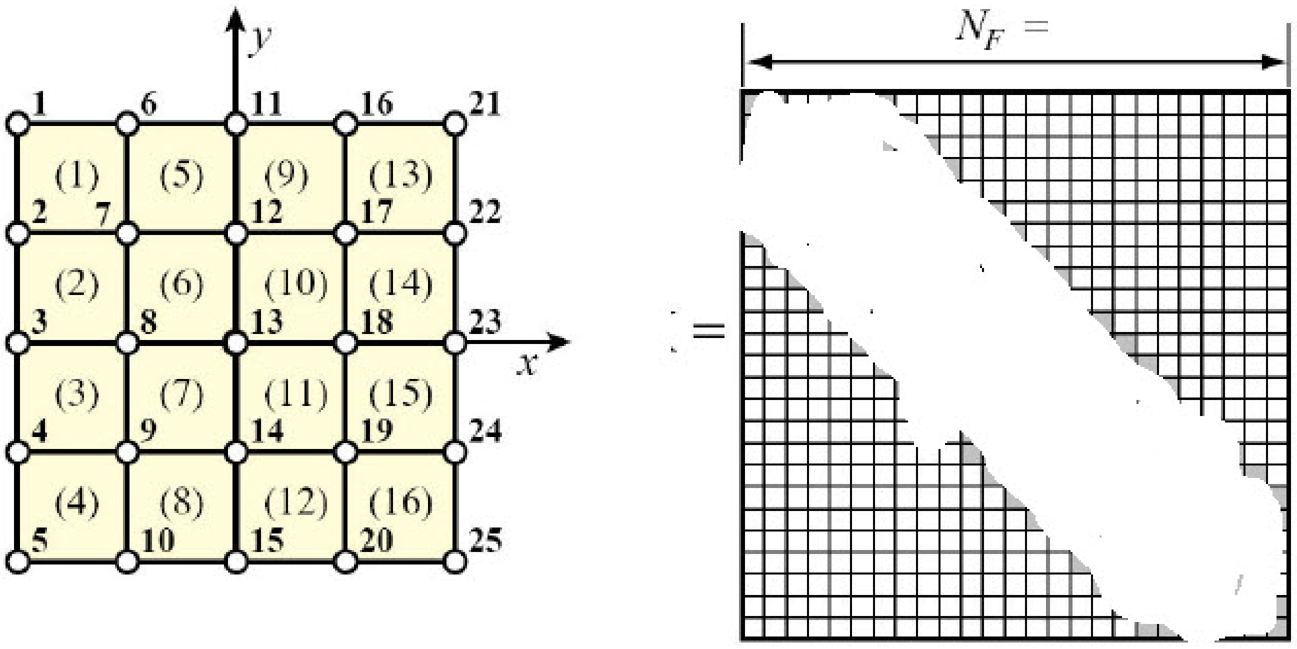
001 LA MATRIZ DE RIGIDEZ GLOBAL ES UNA MATRIZ "EN BANDA" CURSO 2004-5



001 RECURSOS COMPUTACIONALES NECESARIOS SOLUCION ECUACIONES – MATRIZ BANDA CURSO 2004-5

Storage and Solution Times for a Skyline Stiffness Matrix Assuming $B = \sqrt{N}$

Matrix order N	Storage (double prec)	Factor op. units	Factor time workstation/PC	Factor time supercomputer
10^4	8 MB		5 sec	0.05 sec
10^5	240 MB	$10^{10}/2$	8 min	5 sec
10^6	8000 MB	$10^{12}/2$	15 hrs	8 min

time numbers last adjusted in 1998
to get current times divide by 10-20

If a skymatrix of order N can be stored in S memory locations, the ratio $= S/N$ is called the If the entries are, as usual, 8-byte double-precision floating-point numbers, the storage requirement is $8NB$ bytes. The factorization of a skymatrix requires approximately $\frac{1}{2}NB^2$ floating-point operation units. In two-dimensional problems B is of the order of $\sqrt{N}$.

001 CONTENIDOS A TRATAR EN ESTA LECCION CURSO 2004-5

How the Master Stiffness Equations are Stored in a commonly used "skyline" sparse format

How to Mark BC on the Master Stiffness Eqs (if you write your own solver)

The Basic Solution Steps

(Implementation Details will be Skipped since Built-in Mathematica Solver will be used for Demo Programs)

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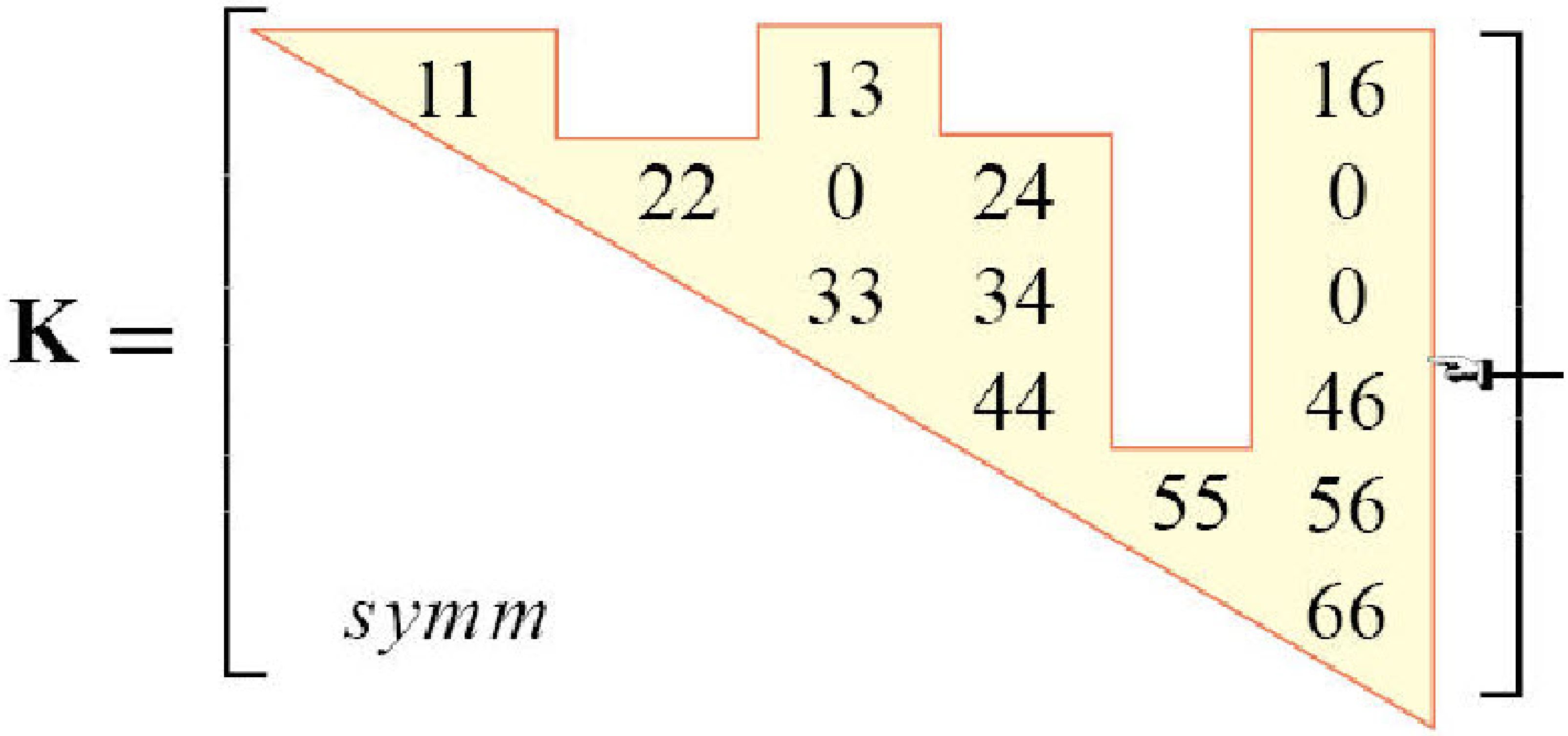
001 **<u>ALMACENAMIENTO</u> TIPO SKYLINE - EJEMPLO** **CURSO 2004-5**

MATRIZ DE RIGIDEZ



VECTOR SKYLINE

s= { }

VECTOR LOCALIZACION TERMINOS DIAGONAL

p= { , , }

VECTOR COMPLETO

S= { p, s }

S={{0,1,2,5,8,9,15},{11,22,13,0,33,24,34,44,55,16,0,0,46,56,66}}

001 **<u>MARCADO CONDICIONES CONTORNO</u> EN DESPLAZAMIENTOS** **CURSO 2004-5**

Equations for which the displacement component is known or prescribed are identified by a ***negative*** diagonal location value. For example, if u_3 and u_5 are prese4cribed displacement components in the sample system,

p : [0, 1, 2, , 8, , 15]

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001 **PASOS BASICOS PROCESO OBTENCION DESPLAZAMIENTOS** **CURSO 2004-5**

FACTORIZACION

$$\mathbf{K} = \mathbf{LDU} = \mathbf{LDL}^T = \mathbf{U}^T\mathbf{DU}$$

where is a unit lower triangular matrix, is a nonsingular diagonal matrix. and **U** and **L** are the transpose of each other. The original matrix is overwritten by the entries of and

SymmSkyMatrixFactor.

SOLUCION

$$\textit{Forward reduction}: \quad \mathbf{Lz} = \mathbf{f},$$

$$\textit{Diagonal scaling}: \quad \mathbf{Dy} = \mathbf{z},$$

$$\textit{Back substitution}: \quad \mathbf{Uu} = \mathbf{y},$$

SymmSkyMatrixVectorSolve.