UNIVERSIDAD POLITECNICA DE VALENCIA
DEPARTAMENTO DE INGENIERIA MECANICA Y DE MATERIALES

ELEMENTOS FINITOS
(E.T.S.I.I.V)

FORMULACION DE ELEMENTOS FINITOS
LECCION 3.- EL PROBLEMA DE TENSION PLANA

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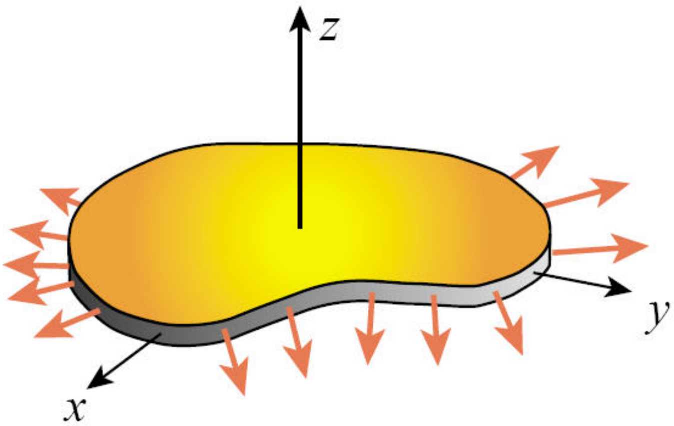
001 ASPECTO DEL PROBLEMA DE TP CURSO 2004-5

Introduction to FEM

Plate in Plane Stress

Thickness dimension or transverse dimension



Inplane dimensions: in x,y **plane**

001 SUPOSICIONES FISICAS DEL PROBLEN TP CURSO 2004-5

Introduction to FEM

Plane Stress Physical Assumptions

Plate is flat and has a symmetry plane (the midplane)

All loads and support conditions are midplane symmetric

Thickness dimension is much smaller than inplane dimensions

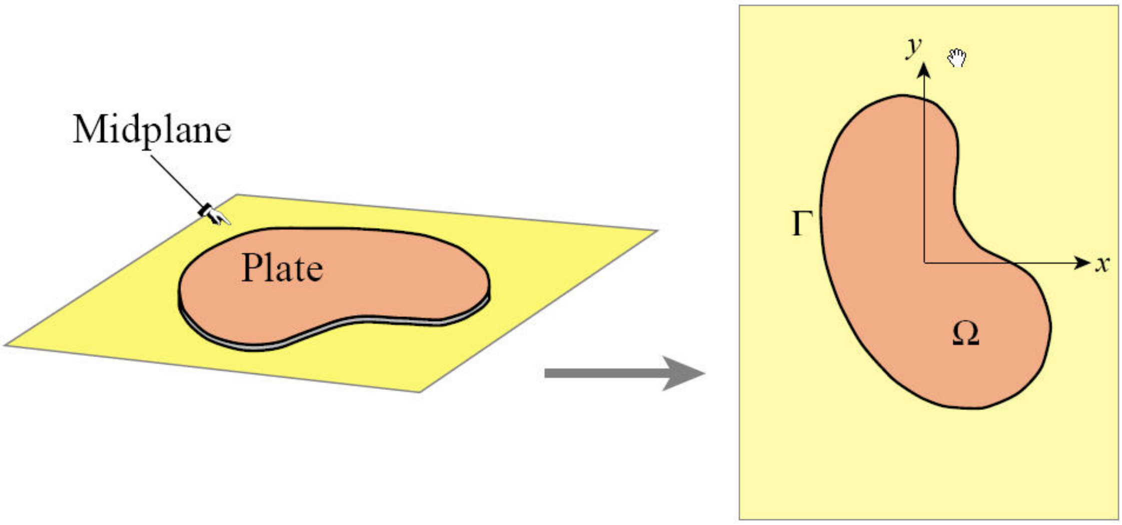
Inplane displacements, strains and stresses uniform through thickness

Transverse stresses σ_{zz}, σ_{xz} **and** σ_{yz} **negligible**

Unessential but used in this course:

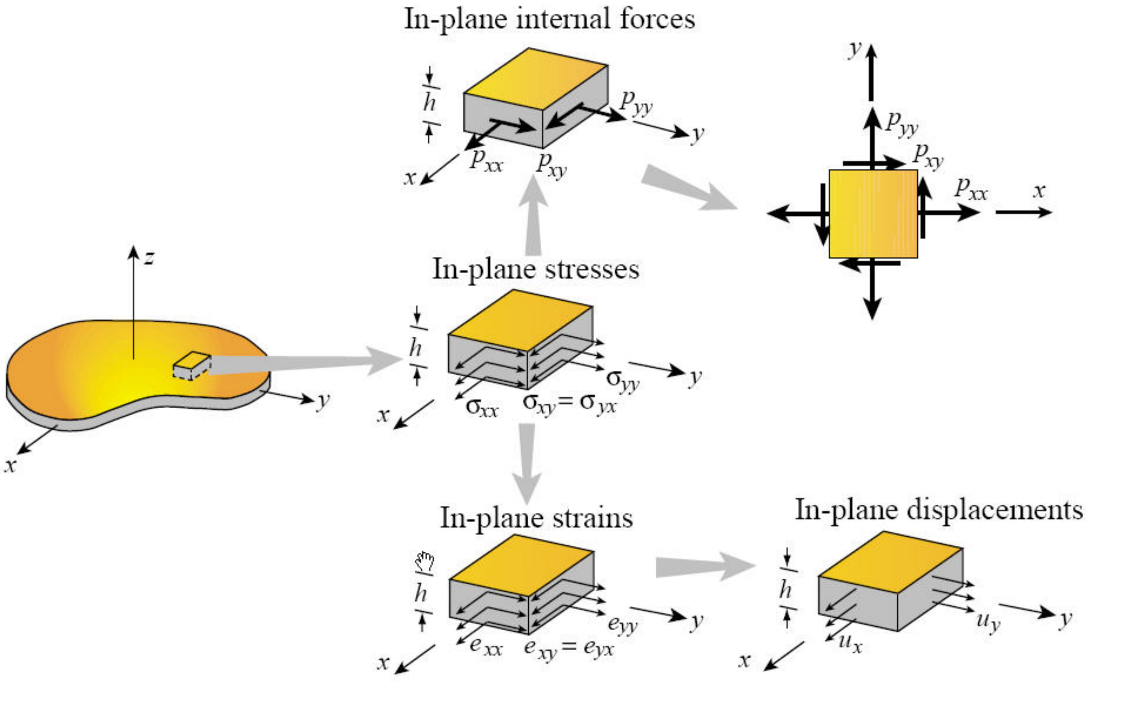
Plate fabricated of homogeneous material through thickness

Mathematical Idealization as a Two Dimensional Problem



Introduction to FEM

Notation for stresses, strains, forces, displacements



The Plane Stress Problem

Given:

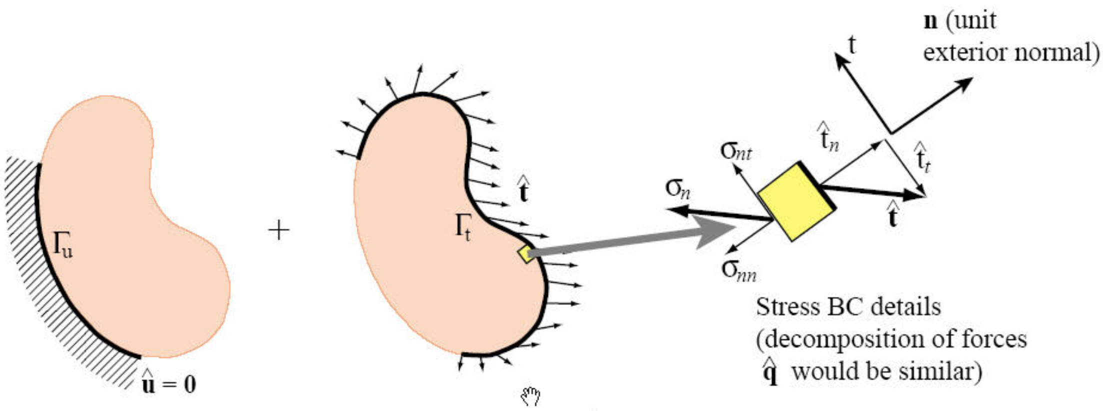
geometry
material properties
wall fabrication (thickness only for homogeneous plates)
applied body forces
boundary conditions:
  prescribed boundary forces or tractions
  prescribed displacements

Find:

inplane displacements
inplane strains
inplane stresses and/or internal forces

Introduction to FEM

Plane Stress Boundary Conditions

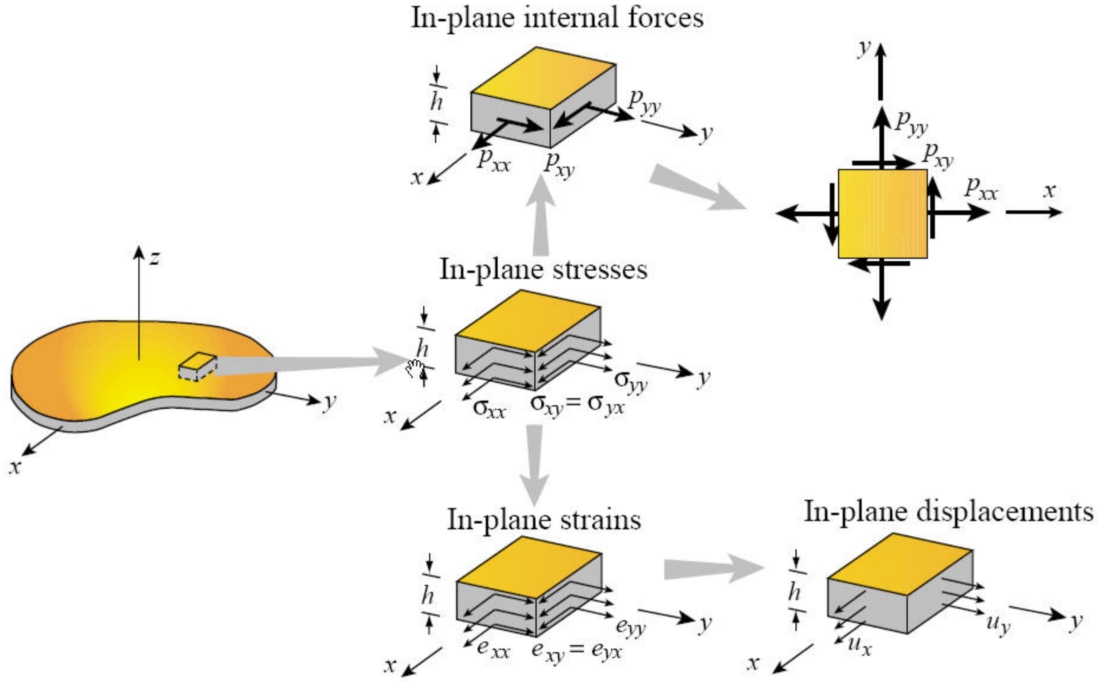


Boundary displacements $\hat{\mathbf{u}}$ are prescribed on Γ_u

Boundary tractions $\hat{\mathbf{t}}$ or boundary forces $\hat{\mathbf{q}}$ are prescribed on Γ_t

Introduction to FEM

Notation for stresses, strains, forces, displacements



Introduction to FEM

Matrix Notation for Internal Fields

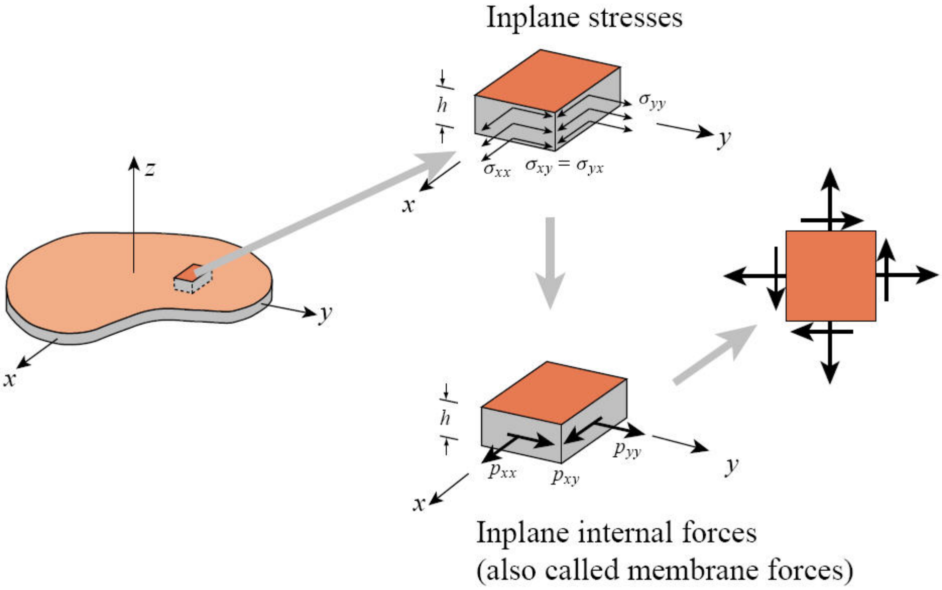
$$\mathbf{u}(x, y) = \begin{bmatrix} u_x(x, y) \\ u_y(x, y) \end{bmatrix}$$ ***displacements***

$$\mathbf{e}(x, y) = \begin{bmatrix} e_{xx}(x, y) \\ e_{yy}(x, y) \\ 2e_{xy}(x, y) \end{bmatrix}$$ ***strains***

$$\boldsymbol{\sigma}(x, y) = \begin{bmatrix} \sigma_{xx}(x, y) \\ \sigma_{yy}(x, y) \\ \sigma_{xy}(x, y) \end{bmatrix}$$ ***stresses***

Introduction to FEM

Inplane Forces are Obtained by Stress Integration Through Thickness



$$p_{xx} = \sigma_{xx} h, \quad p_{yy} = \sigma_{yy} h, \quad p_{xy} = \sigma_{xy} h.$$

Introduction to FEM

Governing Plane Stress Elasticity Equations in Matrix Form

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

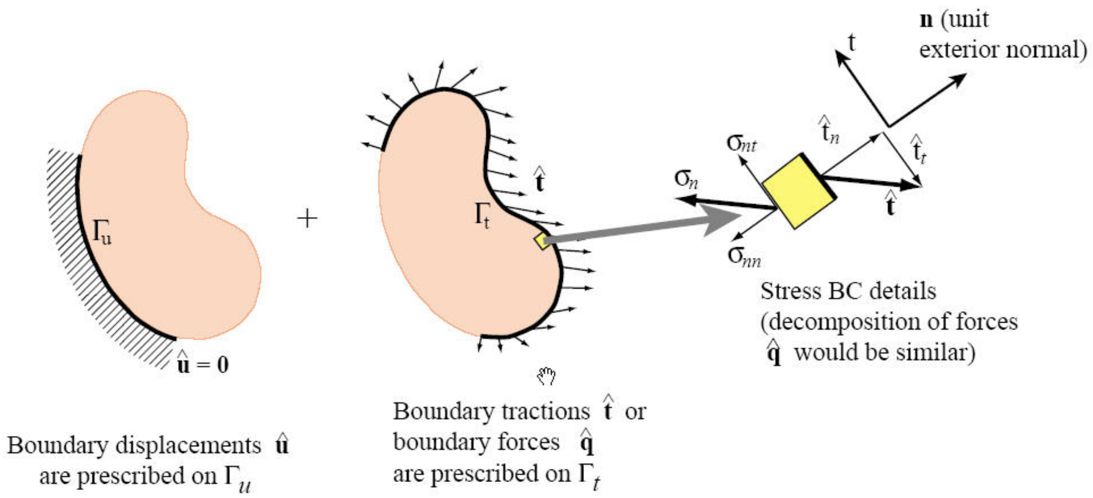
$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

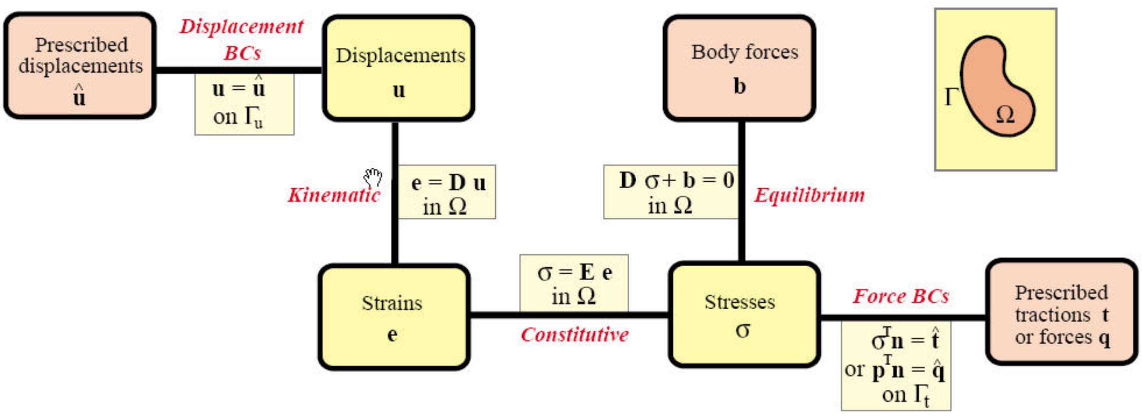
$$\mathbf{e} = \mathbf{D}\mathbf{u} \qquad \boldsymbol{\sigma} = \mathbf{E}\mathbf{e} \qquad \mathbf{D}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$$

Plane Stress Boundary Conditions



$\mathbf{u} = \hat{\mathbf{u}}.$	$\sigma_n = \hat{\mathbf{t}}.$	$\mathbf{p}_n = \hat{\mathbf{q}}.$

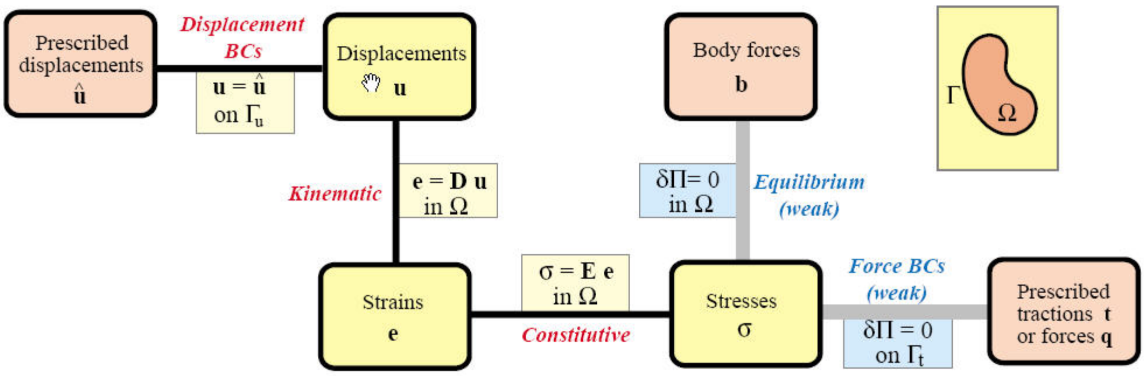
Strong Form Tonti Diagram of Plane Stress Governing Equations



001 DIAGRAMA CONDICIONES “PROMEDIO” CURSO 2004-5

Introduction to FEM

TPE-Based Weak Form Diagram of Plane Stress Governing Equations



001 ENERGIA POTENCIAL TOTAL PLACA EN TP CURSO 2004-5

Introduction to FEM

Total Potential Energy of Plate in Plane Stress

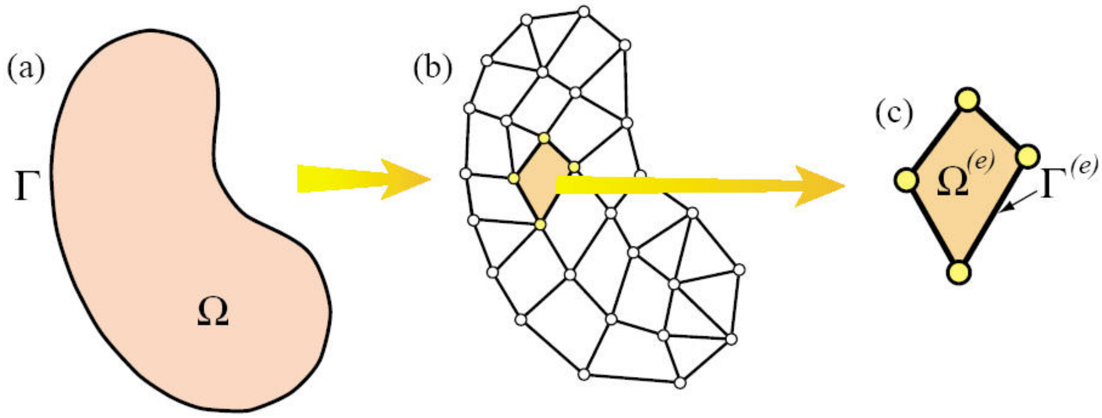
$$\Pi = U - W$$

$$U = \frac{1}{2}\int_{\Omega} h\, \boldsymbol{\sigma}^T \mathbf{e}\, d\Omega = \frac{1}{2}\int_{\Omega} h\, \mathbf{e}^T \mathbf{E}\mathbf{e}\, d\Omega$$

$$W = \int_{\Omega} h\, \mathbf{u}^T \mathbf{b}\, d\Omega + \int_{\Gamma_t} h\, \mathbf{u}^T \hat{\mathbf{t}}\, d\Gamma$$

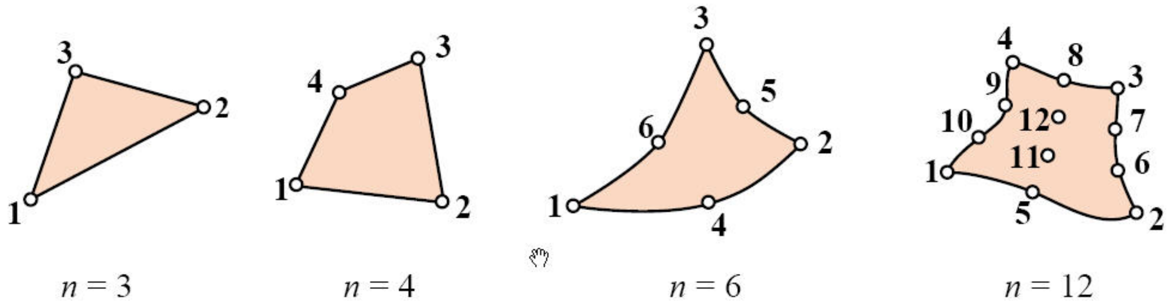
Introduction to FEM

Discretization into Plane Stress Finite Elements



Introduction to FEM

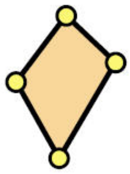
Plane Stress Element Geometries and Node Configurations



$$\mathbf{u}^{(e)} = [\, u_{x1} \quad u_{y1} \quad u_{x2} \quad \ldots \quad u_{xn} \quad u_{yn} \,]^T .$$

Introduction to FEM

Constructing a Displacement Assumed Element



n nodes, n=4 in figure

Node displacement vector:

$$\mathbf{u}^{(e)} = [\, u_{x1} \quad u_{y1} \quad u_{x2} \quad \ldots \quad u_{xn} \quad u_{yn} \,]^T$$

Displacement interpolation

$$\mathbf{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} N_1^{(e)} & 0 & N_2^{(e)} & 0 & \ldots & N_n^{(e)} & 0 \\ 0 & N_1^{(e)} & 0 & N_2^{(e)} & \ldots & 0 & N_n^{(e)} \end{bmatrix} \mathbf{u}^{(e)}$$

$$= \mathbf{N}\ \mathbf{u}^{(e)}$$

N **is called the shape function matrix**

Introduction to FEM

Requirements on Finite Element Shape Functions

Interpolation Conditions:

N_i **takes on value 1 at node i, 0 at all other nodes**

Continuity (intra- and inter-element) and Completeness Conditions

are covered later in the course (Chs. 18-19)

001	OBTENCION DE LAS DEFORMACIONES	CURSO 2004-5

Introduction to FEM

Element Construction (cont'd)

Differentiate the displacement interpolation wrt *x,y* **to get the strain-displacement relation**

$$\mathbf{e}(x,y) = \begin{bmatrix} \frac{\partial N_1^{(e)}}{\partial x} & 0 & \frac{\partial N_2^{(e)}}{\partial x} & 0 & \dots & \frac{\partial N_n^{(e)}}{\partial x} & 0 \\ 0 & \frac{\partial N_1^{(e)}}{\partial y} & 0 & \frac{\partial N_2^{(e)}}{\partial y} & \dots & 0 & \frac{\partial N_n^{(e)}}{\partial y} \\ \frac{\partial N_1^{(e)}}{\partial y} & \frac{\partial N_1^{(e)}}{\partial x} & \frac{\partial N_2^{(e)}}{\partial y} & \frac{\partial N_2^{(e)}}{\partial x} & \dots & \frac{\partial N_n^{(e)}}{\partial y} & \frac{\partial N_n^{(e)}}{\partial x} \end{bmatrix} \mathbf{u}^{(e)} = \mathbf{B}\mathbf{u}^{(e)}$$

B is called the strain-displacement matrix

$$\mathbf{B} = \mathbf{D}\mathbf{N}^{(e)}$$

001	OBTENCION DE LAS TENSIONES	CURSO 2004-5

$$\boldsymbol{\sigma} = \mathbf{E}\,\mathbf{e} = \mathbf{E}\mathbf{B}\mathbf{u}^{(e)}$$

001	PROCEDIMIENTO DE OBTENER LA ECUACION DEL ELEMENTO FINITO	CURSO 2004-5

$$\delta\Pi^{(e)} = \delta U^{(e)} - \delta W^{(e)} = 0.$$

$$U^{(e)} = \frac{1}{2}\int_{\Omega^{(e)}} h\,\boldsymbol{\sigma}^T\mathbf{e}\,d\Omega^{(e)} = \frac{1}{2}\int_{\Omega^{(e)}} h\,\mathbf{e}^T\mathbf{E}\mathbf{e}\,d\Omega^{(e)}$$

$$W^{(e)} = \int_{\Omega^{(e)}} h\,\mathbf{u}^T\mathbf{b}\,d\Omega^{(e)} + \int_{\Gamma^{(e)}} h\,\mathbf{u}^T\,\hat{\mathbf{t}}\,d\Gamma^{(e)}$$

$$\mathbf{u} = \mathbf{N}\mathbf{u}^{(e)}$$

$$\mathbf{e} = \mathbf{B}\mathbf{u}^{(e)}$$

$$\boldsymbol{\sigma} = \mathbf{E}\mathbf{e}$$

Element Construction (cont'd)

Element total potential energy

$$\Pi^{(e)} = \frac{1}{2}\mathbf{u}^{(e)T}\mathbf{K}^{(e)}\mathbf{u}^{(e)} - \mathbf{u}^{(e)T}\mathbf{f}^{(e)}$$

Element stiffness matrix

$$\mathbf{K}^{(e)} = \int_{\Omega^{(e)}} h\,\mathbf{B}^T\mathbf{E}\mathbf{B}\,d\Omega^{(e)}$$

Consistent node force vector

$$\mathbf{f}^{(e)} = \int_{\Omega^{(e)}} h\,\mathbf{N}^T\mathbf{b}\,d\Omega^{(e)} + \int_{\Gamma^{(e)}} h\,\mathbf{N}^T\hat{\mathbf{t}}\,d\Gamma^{(e)}$$

body force **surface force**

The calculation of the entries of $\mathbf{K}^{(e)}$ and $\mathbf{f}^{(e)}$ for several elements of historical or practical interest is described in subsequent Chapters.

001 **EJERCICIO 1** CURSO 2004-5

EXERCISE 14.1

[A:25] Suppose that the structural material is isotropic, with elastic modulus E and Poisson's ratio ν. The in-plane stress-strain relations for plane stress ($\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$) and plane strain ($e_{zz} = e_{xz} = e_{yz} = 0$) as given in any textbook on elasticity, are

$$\text{plane stress:} \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix},$$

$$\text{plane strain:} \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}. \qquad \text{(E14.1)}$$

Show that the constitutive matrix of plane strain can be formally obtained by replacing E by a fictitious modulus E^* and ν by a fictitious Poisson's ratio ν^* in the plane stress constitutive matrix and suppressing the stars. Find the expression of E^* and ν^* in terms of E and ν. This device permits "reusing" a plane stress FEM program to do plane strain, as long as the material is isotropic.

001 **EJERCICIO 2** CURSO 2004-5

EXERCISE 14.2

[A:25] In the finite element formulation of near incompressible isotropic materials (as well as plasticity and viscoelasticity) it is convenient to use the so-called *Lamé constants* λ and μ instead of E and ν in the constitutive equations. Both λ and μ have the physical dimension of stress and are related to E and ν by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = G = \frac{E}{2(1+\nu)}. \qquad \text{(E14.2)}$$

Conversely

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}. \qquad \text{(E14.3)}$$

Substitute (E14.3) into (E14.1) to express the two stress-strain matrices in terms of λ and μ. Then split the stress-strain matrix $\mathbf{E}$ of plane strain as

$$\mathbf{E} = \mathbf{E}_\mu + \mathbf{E}_\lambda \qquad \text{(E14.4)}$$

in which $\mathbf{E}_\mu$ and $\mathbf{E}_\lambda$ contain only μ and λ, respectively, with $\mathbf{E}_\mu$ diagonal and $E_{\lambda 33} = 0$. This is the Lamé or $\{\lambda, \mu\}$ splitting of the plane strain constitutive equations, which leads to the so-called **B**-bar formulation of near-incompressible finite elements.[3] Express $\mathbf{E}_\mu$ and $\mathbf{E}_\lambda$ also in terms of E and ν.

For the plane stress case perform a similar splitting in which where $\mathbf{E}_\lambda$ contains only $\bar{\lambda} = 2\lambda\mu/(\lambda + 2\mu)$ with $E_{\lambda 33} = 0$, and $\mathbf{E}_\mu$ is a diagonal matrix function of μ and $\bar{\lambda}$.[4] Express $\mathbf{E}_\mu$ and $\mathbf{E}_\lambda$ also in terms of E and ν.

EXERCISE 14.5

[A:25=5+5+15] A plate is in linearly elastic plane stress. It is shown in courses in elasticity that the internal strain energy density stored per unit volume is

$$\mathcal{U} = \tfrac{1}{2}(\sigma_{xx}e_{xx} + \sigma_{yy}e_{yy} + \sigma_{xy}e_{xy} + \sigma_{yx}e_{yx}) = \tfrac{1}{2}(\sigma_{xx}e_{xx} + \sigma_{yy}e_{yy} + 2\sigma_{xy}e_{xy}). \qquad \text{(E14.5)}$$

(a) Show that (E14.5) can be written in terms of strains only as

$$\mathcal{U} = \tfrac{1}{2}\mathbf{e}^T\,\mathbf{E}\,\mathbf{e}, \qquad \text{(E14.6)}$$

and hence justify (14.13).

(b) Show that (E14.5) can be written in terms of stresses only as

$$\mathcal{U} = \tfrac{1}{2}\boldsymbol{\sigma}^T\,\mathbf{C}\,\boldsymbol{\sigma}, \qquad \text{(E14.7)}$$

where $\mathbf{C} = \mathbf{E}^{-1}$ is the elastic compliance (strain-stress) matrix.

(c) Suppose you want to write (E14.5) in terms of the extensional strains $\{e_{xx}, e_{yy}\}$ and of the shear stress $\sigma_{xy} = \sigma_{yx}$. This is known as a mixed representation. Show that

$$\mathcal{U} = \tfrac{1}{2}\begin{bmatrix} e_{xx} \\ e_{yy} \\ \sigma_{xy} \end{bmatrix}^T \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ \sigma_{xy} \end{bmatrix}, \qquad \text{(E14.8)}$$

and explain how the entries A_{ij} can be calculated[5] in terms of the elastic moduli E_{ij}.